Pass-Join: A Partition-based Method for Similarity Joins

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ABSTRACT

As an essential operation in data cleaning, the similarity join has attracted considerable attention from the database community. In this paper, we study string similarity joins with edit-distance constraints, which find *similar* string pairs from two large sets of strings whose edit distance is within a given threshold. Existing algorithms are efficient either for short strings or for long strings, and there is no algorithm that can efficiently and adaptively support both short strings and long strings. To address this problem, we propose a partition-based method called Pass-Join. Pass-Join partitions a string into a set of segments and creates inverted indices for the segments. Then for each string, Pass-Join selects some of its substrings and uses the selected substrings to find candidate pairs using the inverted indices. We devise efficient techniques to select the substrings and prove that our method can minimize the number of selected substrings. We develop novel pruning techniques to efficiently verify the candidate pairs. Experimental results show that our algorithms are efficient for both short strings and long strings, and outperform state-of-the-art methods on real datasets.

1. INTRODUCTION

A string similarity join between two sets of strings finds all *similar* string pairs from the two sets. For example, consider two sets of strings {vldb, sigmod, ...} and {pvldb, icde, ...}. We want to find all similar pairs, e.g., ⟨vldb, pvldb⟩. Many similarity functions have been proposed to quantify the similarity between two strings, such as jaccard similarity, cosine similarity, and edit distance. In this paper, we study string similarity joins with edit-distance constraints, which, given two sets of strings, find all *similar* string pairs from the two sets, such that the edit distance between each string pair is within a given threshold. The string similarity join is an essential operation in many applications, such as data integration and cleaning, near duplicate object detection and elimination, and collaborative filtering.

Existing methods to address this problem can be broadly classified into two categories. The first one uses a *filter-and-refine* framework, such as Part-Enum [2], All-Pairs-Ed [3], ED-Join [23]. In the *filter* step, they generate signatures for each string and use the signatures to generate candidate pairs. In the *refine* step, they verify the candidate pairs to generate the final results. However, these approaches are inefficient for the datasets with short strings (e.g., person names and locations) [20]. The main reason is that they cannot select high-quality signatures for short strings and will generate large numbers of candidates which need to be further verified. The second method (Trie-Join [20]) adopts a trie-based framework, which uses a trie structure to share prefixes and utilizes prefix pruning to improve the performance. However Trie-Join is inefficient for long strings (e.g., paper titles and abstracts). This is because long strings have a small number of shared prefixes.

If a system wants to support both short strings and long strings, we have to implement and maintain two separate codes, and tune many parameters to select the best method. To alleviate this problem, it calls for an adaptive method which can efficiently support both short strings and long strings. In this paper we propose a partition-based method to address this problem. We devise a partition scheme to partition a string into a set of segments and prove that if a string s is similar to string r, s must have a substring which matches a segment of r. Based on this observation, we propose a *pa*rtition-based framework for *s*tring *s*imilarity *joins*, called Pass-Join. Pass-Join creates inverted indices for the segments. For each string s, we select some of its substrings, and search for the selected substrings in the inverted indices. If a selected substring appears in the inverted index, each string r on the inverted list of this substring (i.e., r contains the substring) may be similar to s, and we take r and s as a candidate pair. Next we verify the pair to generate the final answers. We develop effective techniques to select substrings and prove that our method can minimize the number of selected substrings. We devise novel pruning techniques to verify candidate pairs. To summarize, we make the following contributions.

(1) We devise a partition scheme to partition strings into a set of segments. Using the partition scheme, we propose a partition-based framework to facilitate similarity joins.

(2) We develop novel techniques to select substrings and use them to generate candidate pairs. We prove that our method can minimize the number of selected substrings.

(3) We propose an extension-based method to efficiently verify a candidate pair, and develop pruning techniques and

early-termination techniques to improve the performance.

(4) We have conducted an extensive set of experiments. Experimental results show that our algorithms are very efficient for both short strings and long strings, and outperform state-of-the-art methods on real datasets.

The rest of this paper is organized as follows. We formalize our problem in Section 2. Section 3 introduces our partition-based framework. We propose to effectively select substrings in Section 4 and develop novel techniques to efficiently verify candidates in Section 5. Experimental results are provided in Section 6. We review related work in Section 7 and make a conclusion in Section 8.

2. PROBLEM FORMULATION

Given two collections of strings, a similarity join finds all *similar* string pairs from the two collections. In this paper, we use edit distance to quantify the similarity between two strings. Formally, the edit distance between two strings r and s, denoted by $\text{ED}(r, s)$, is the minimum number of single-character edit operations (i.e., insertion, deletion, and substitution) needed to transform r to s. For example, ED(“kaushic chaduri”, “kaushuk chadhui”) = 4.

In this paper two strings are *similar* if their edit distance is not larger than a specified edit-distance threshold τ. We formalize the problem of string similarity joins as follows.

DEFINITION 1 (STRING SIMILARITY JOINS). *Given two sets of strings $\mathcal{R}$ and $\mathcal{S}$ and an edit-distance threshold τ, a similarity join finds all similar string pairs $\langle r, s\rangle \in \mathcal{R} \times \mathcal{S}$ such that $\text{ED}(r, s) \le \tau$.*

Without loss of generality, we focus on self join in this paper, that is $\mathcal{R} = \mathcal{S}$. We will discuss how to join two distinct sets ($\mathcal{R} \neq \mathcal{S}$) in Section 3.

For example, consider the strings in Table 1(a). Suppose threshold τ=3. ⟨“kaushik chakrab”, “caushik chakrabar”⟩ is a similar pair as their edit distance is not larger than τ.

Table 1: A set of strings

(a) Strings

Strings
avataresha
caushik chakrabar
kaushic chaduri
kaushik chakrab
kaushuk chadhui
vankatesh

(b) Sorted strings

ID	Strings	Length
s_1	vankatesh	9
s_2	avataresha	10
s_3	kaushic chaduri	15
s_4	kaushik chakrab	15
s_5	kaushuk chadhui	15
s_6	caushik chakrabar	17

3. PARTITION-BASED SIMILARITY JOINS

We first introduce a partition scheme to partition a string into several disjoint segments (Section 3.1), and then propose a partition-based framework (Section 3.2).

3.1 Partition Scheme

Given a string s, we partition it into $\tau + 1$ disjoint segments, and the length of each segment is not smaller than one*. For example, consider string s_1=“vankatesh”. Suppose $\tau = 3$. We have multiple ways to partition s_1 into $\tau + 1 = 4$ segments, such as {“va”,“nk”,“at”, “esh”}.

Consider two strings r and s. If s has no substring that matches a segment of r, s cannot be similar to r based on the pigeonhole principle as stated in Lemma 1. Due to space constraints, we refer readers to our technical [16]. In other words, if s is similar to r, s must contain a substring matching a segment of r. For example, consider strings in Table 2. Suppose τ=3. String s_1=“vankatesh” has four segments {“va”, “nk”, “at”, “esh”}. As s_3, s_4, s_5, s_6 have no substrings matching segments of s_1, they are not similar to s_1.

*The length of string $s(|s|)$ should be larger than τ, i.e., $|s| \ge \tau + 1$.

LEMMA 1. *Given a string r with $\tau + 1$ segments and a string s, if s is similar to r within threshold τ, s must contain a substring which matches a segment of r.*

Given a string, there could be many strategies to partition the string into $\tau+1$ segments. A good partition strategy can reduce the number of candidate pairs and thus improve the performance. Intuitively, the shorter a segment of r is, the higher probability the segment appears in other strings, and the more strings will be taken as r's candidates, thus the pruning power is lower. Based on this observation, we do not want to keep short segments in the partition. In other words, each segment should have nearly the same length. Accordingly we propose an *even-partition* scheme. Consider a string s with length $|s|$. In even partition, each segment has a length of $\lfloor \frac{|s|}{\tau+1} \rfloor$ or $\lceil \frac{|s|}{\tau+1} \rceil$, thus the maximal length difference between two segments is 1. Let $k = |s| - \lfloor \frac{|s|}{\tau+1} \rfloor * (\tau + 1)$. In even partition, the last k segments have length $\lceil \frac{|s|}{\tau+1} \rceil$, and the first $\tau + 1 - k$ ones have length $\lfloor \frac{|s|}{\tau+1} \rfloor$. For example, consider s_1=“vankatesh” and suppose $\tau = 3$. We have $k = 1$. s_1 has four segments {“va”,“nk”,“at”, “esh”}.

Although we can devise other partition schemes, it is time consuming to select a good partition strategy. Note that the time for selecting a partition strategy should be included in the similarity join time. In this paper we use the even-partition scheme and leave the problem of selecting good partition strategies as a future work. Note that our proposed techniques can be extended to other partition strategies.

3.2 Partition-based Framework

We have an observation that if a strings s does not have a substring that matches a segment of r, we can prune the pair $\langle s, r\rangle$. We can use this feature to prune large numbers of dissimilar pairs. To this end, we propose a *pa*rtition-based framework for *s*tring *s*imilarity *joins*, called PASS-JOIN. Figure 2 illustrates our framework.

For ease of presentation, we first introduce some notations. Let $\mathcal{S}_l$ denote the set of strings with length l and $\mathcal{S}_l^i$ denote the set of the i-th segments of strings in $\mathcal{S}_l$. We build an inverted index for each $\mathcal{S}_l^i$, denoted by $\mathcal{L}_l^i$. Given an i-th segment w, let $\mathcal{L}_l^i(w)$ denote the inverted list of segment w, i.e., the set of strings whose i-th segments are w. PASS-JOIN uses the inverted indices to do similarity joins as follows.

PASS-JOIN first sorts strings based on their lengths in ascending order. For the strings with the same length, it sorts them in alphabetical order. Then PASS-JOIN visits strings in order. Consider the current string s with length $|s|$. PASS-JOIN finds s's similar strings among the visited strings using the inverted indices. To efficiently find such strings, we create indices only for visited strings to avoid enumerating a string pair twice. Based on length filtering [7], we check whether the strings in $\mathcal{L}_l^i$ $(|s| - \tau \le l \le |s|, 1 \le i \le \tau + 1)$ are similar to s. Without loss of generality, consider inverted index $\mathcal{L}_l^i$. PASS-JOIN finds s's similar strings in $\mathcal{L}_l^i$ as follows.

- SUBSTRING SELECTION: If s is similar to a string in $\mathcal{L}_l^i$, s should contain a substring which matches a segment in $\mathcal{L}_l^i$. A straightforward method enumerates all of

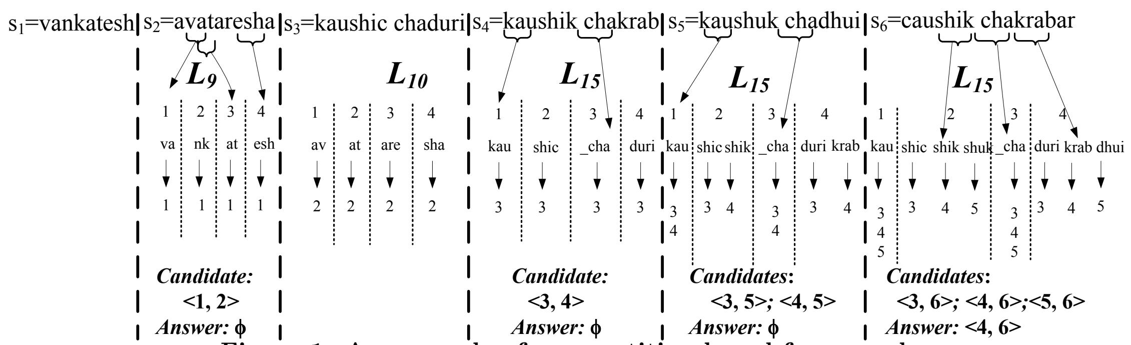


Figure 1: An example of our partition-based framework

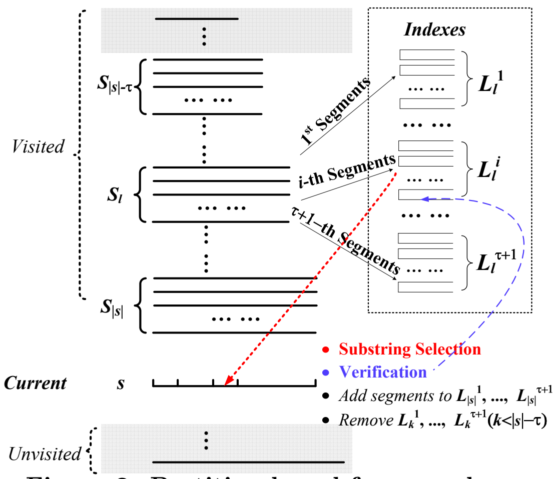


Figure 2: Partition-based framework

s's substrings, and for each substring checks whether it appears in $\mathcal{L}_l^i$. Actually we do not need to consider all substrings of s. Instead we only select some substrings (denoted by $\mathcal{W}(s, \mathcal{L}_l^i)$) and use the selected substrings to find similar pairs. We discuss how to generate $\mathcal{W}(s, \mathcal{L}_l^i)$ in Section 4. For each selected substring $w \in \mathcal{W}(s, \mathcal{L}_l^i)$, we check whether it appears in $\mathcal{L}_l^i$. If so, for each $r \in \mathcal{L}_l^i(w)$, $\langle r, s\rangle$ is a candidate pair.

- Verification: To verify whether a candidate pair $\langle r, s\rangle$ is an answer, a straightforward method computes their real edit distance. However this method is rather expensive, and we develop effective techniques to do efficient verification in Section 5.

After finding similar strings for s, we partition s into $\tau+1$ segments and insert the segments into inverted index $\mathcal{L}_{|s|}^i (1 \le i \le \tau+1)$. Then we visit strings after s and iteratively we can find all similar pairs. Note that we can remove the inverted index $\mathcal{L}_k^i$ for $k < |s| - \tau$. Thus we maintain at most $(\tau+1)^2$ inverted indices $\mathcal{L}_l^i$ for $|s|-\tau \le l \le |s|$ and $1 \le i \le \tau+1$.

To join two distinct sets $\mathcal{R}$ and $\mathcal{S}$, we first sort the strings in the two sets respectively. Then we index the segments of strings in a set, e.g., $\mathcal{S}$. Next we visit strings of $\mathcal{R}$ in order. For each string $r \in \mathcal{R}$ with length $|r|$, we use the inverted indices of strings in $\mathcal{S}$ with lengths between $[|r|-\tau, |r|+\tau]$ to find similar pairs. We can remove the indices for strings with lengths smaller than $|r|-\tau$. In this paper we focus on the case that the index can be fit in the memory. We leave dealing with a very large dataset as a future work.

For example, consider strings in Table 1. Suppose $\tau = 3$. We find similar pairs as follows (Figure 1). For the first string s_1 = "`vankatesh`", we partition it into $\tau+1$ segments and insert the segments into the inverted indices for

Algorithm 1: Pass-Join $(\mathcal{S}, \tau)$

```
Input: $\mathcal{S}$: A collection of strings
       $\tau$: A given edit-distance threshold
Output: $\mathcal{A} = \{(s \in \mathcal{S}, r \in \mathcal{S}) \mid \text{ED}(s, r) \le \tau\}$
1 begin
2   Sort $\mathcal{S}$ first by string length and second in alphabetical order;
3   for $s \in \mathcal{S}$ do
4     for $\mathcal{L}_l^i$ $(|s|-\tau \le l \le |s|, 1 \le i \le \tau+1)$ do
5       $\mathcal{W}(s, \mathcal{L}_l^i)$ = SubstringSelection$(s, \mathcal{L}_l^i)$;
6       for $w \in \mathcal{W}(s, \mathcal{L}_l^i)$ do
7         if $w$ is in $\mathcal{L}_l^i$ then
            Verification$(s, \mathcal{L}_l^i(w), \tau)$;
8     Partition $s$ and add its segments into $\mathcal{L}_{|s|}^i$;
9 end
```

Function SubstringSelection$(s, \mathcal{L}_l^i)$

```
Input: $s$: A string; $\mathcal{L}_l^i$: Inverted index
Output: $\mathcal{W}(s, \mathcal{L}_l^i)$: Selected substrings
1 begin
2   $\mathcal{W}(s, \mathcal{L}_l^i) = \{w \mid w$ is a substring of $s\}$;
3 end
```

Function Verification$(s, \mathcal{L}_l^i(w), \tau)$

```
Input: $s$: A string; $\mathcal{L}_l^i(w)$: Inverted list; $\tau$: Threshold
Output: $\mathcal{A} = \{(s \in \mathcal{S}, r \in \mathcal{S}) \mid \text{ED}(s, r) \le \tau\}$
1 begin
2   for $r \in \mathcal{L}_l^i(w)$ do
3     if ED$(s, r) \le \tau$ then $\mathcal{A} \leftarrow \langle s, r\rangle$;
4 end
```

Figure 3: Pass-Join algorithm

strings with length 9, i.e., $\mathcal{L}_9^1, \mathcal{L}_9^2, \mathcal{L}_9^3, \mathcal{L}_9^4$. Next for s_2 = "`avataresha`", we enumerate its substrings, and check whether each substring appears in $\mathcal{L}_{|s_2|-\tau}^i, \cdots, \mathcal{L}_{|s_2|}^i (1 \le i \le \tau+1)$. Here we find "`va`" in $\mathcal{L}_9^1$, "`at`" in $\mathcal{L}_9^3$, and "`esh`" in $\mathcal{L}_9^4$. For segment "`va`", as $\mathcal{L}_9^1(\texttt{va}) = \{s_1\}$. The pair $\langle s_2, s_1\rangle$ is a candidate pair. We verify the pair and it is not an answer as the edit distance is larger than τ. Next we partition s_2 into four segments and insert them into $\mathcal{L}_{|s_2|}^1, \mathcal{L}_{|s_2|}^2, \mathcal{L}_{|s_2|}^3, \mathcal{L}_{|s_2|}^4$. Similarly we repeat the above steps and find all similar pairs.

We give the pseudo-code of our algorithm in Figure 3. Pass-Join sorts strings first by length and then in alphabetical order (line 2). Then, Pass-Join visits each string in sorted order (line 3). For each inverted index $\mathcal{L}_l^i (|s|-\tau \le l \le |s|, 1 \le i \le \tau+1)$, Pass-Join selects the substrings of s (line 4) and checks whether each selected substring w is in $\mathcal{L}_l^i$ (line 5). If yes, for any string r in the inverted list of w

in $\mathcal{L}_l^i$, i.e., $\mathcal{L}_l^i(w)$, the string pair $\langle r, s\rangle$ is a candidate pair. Pass-Join verifies the pair (line 7). Finally, Pass-Join partitions s into $\tau+1$ segments, and inserts the segments into the inverted index $\mathcal{L}_{|s|}^i(1 \le i \le \tau+1)$ (line 8). Here function SubstringSelection selects all substrings and function Verification computes the real edit distance of two strings to verify the candidates using dynamic-programming algorithm. To improve the performance, we propose effective techniques to improve the substring-selection step in Section 4 and the verification step in Section 5.

Complexity: We first analyze the space complexity. Our indexing structure includes segments and inverted lists of segments. We first give the space complexity of segments. For each string in $\mathcal{S}_l$ we generate $\tau+1$ segments. Thus the number of segments is at most $(\tau+1)\times|\mathcal{S}_l|$, where $|\mathcal{S}_l|$ is the number strings in $\mathcal{S}_l$. As we can use an integer to encode a segment, the space complexity of segments is

$$\mathcal{O}\Big(\max_{l_{min}\le j\le l_{max}} \sum_{l=j-\tau}^{j} (\tau+1)\times|\mathcal{S}_l|\Big),$$

where l_{min} and l_{max} respectively denote the minimal string length and the maximal string length.

Next we give the complexity of inverted lists. For each string in $\mathcal{S}_l$, as the i-th segment of the string corresponds to an element in $\mathcal{L}_l^i$, $|\mathcal{S}_l| = |\mathcal{L}_l^i|$. The space complexity of inverted lists(i.e., the sum of the lengths of inverted lists) is

$$\mathcal{O}\Big(\max_{l_{min}\le j\le l_{max}} \sum_{l=j-\tau}^{j}\sum_{i=1}^{\tau+1}|\mathcal{L}_l^i| = \max_{l_{min}\le j\le l_{max}} \sum_{l=j-\tau}^{j} (\tau+1)\times|\mathcal{S}_l|\Big).$$

Then we give the time complexity. To sort the strings, we can first group the strings based on lengths and then sort strings in each group. Thus the sort complexity is $\mathcal{O}\big(\sum_{l_{min}\le l\le l_{max}}|\mathcal{S}_l|log(|\mathcal{S}_l|)\big)$. For each string s, we select its substring set $\mathcal{W}(s,\mathcal{L}_l^i)$ for $|s|-\tau\le l\le|s|, 1\le i\le\tau+1$. The selection complexity is $\mathcal{O}\Big(\sum_{s\in\mathcal{S}}\sum_{l=|s|-\tau}^{|s|}\sum_{i=1}^{\tau+1}\mathcal{X}(s,\mathcal{L}_l^i)\Big)$, where $\mathcal{X}(s,\mathcal{L}_l^i)$ is the selection time complexity for $\mathcal{W}(s,\mathcal{L}_l^i)$, which is $\mathcal{O}(\tau)$ (Section 4). The selection complexity is $\mathcal{O}\big(\tau^3|\mathcal{S}|\big)$. For each substring $w\in\mathcal{W}(s,\mathcal{L}_l^i)$, we verify whether strings in $\mathcal{L}_l^i(w)$ are similar to s. The verification complexity is $\mathcal{O}\Big(\sum_{s\in\mathcal{S}}\sum_{l=|s|-\tau}^{|s|}\sum_{i=1}^{\tau+1}\sum_{w\in\mathcal{W}(s,\mathcal{L}_l^i)}\sum_{r\in\mathcal{L}_l^i(w)}\mathcal{V}(s,r)\Big)$, where $\mathcal{V}(s,r)$ is the complexity for verifying $\langle s,r\rangle$, which is $\mathcal{O}(\tau*\min(|s|,|r|))$(Section 5). In the paper we propose to reduce the size of $\mathcal{W}(s,\mathcal{L}_l^i)$ and improve the verification cost $\mathcal{V}(s,r)$.

4. IMPROVING SUBSTRING SELECTION

For any string $s\in\mathcal{S}$ and a length l ($|s|-\tau\le l\le|s|$), we select a substring set $\mathcal{W}(s,l)=\cup_{i=1}^{\tau+1}\mathcal{W}(s,\mathcal{L}_l^i)$ of s and use substrings in $\mathcal{W}(s,l)$ to find the candidates of s. We need to guarantee completeness of the method using $\mathcal{W}(s,l)$ to find candidate pairs. That is any similar pair must be found as a candidate pair. Next we give the formal definition.

Definition 2 (Completeness). *A substring selection method satisfies completeness, if for any string s and a length $l(|s|-\tau\le l\le|s|)$, $\forall$ r with length l which is similar to s and visited before s, r must have an i-th segment r_m which matches a substring $s_m\in\mathcal{W}(s,\mathcal{L}_l^i)$ where $1\le i\le\tau+1$.*

A straightforward method is to add all substrings of s into $\mathcal{W}(s,l)$. As s has $|s|-i+1$ substrings with length i, the total number of s's substrings is $\sum_{i=1}^{|s|}(|s|-i+1)=\frac{|s|*(|s|+1)}{2}$. For long strings, there are large numbers of substrings and it is rather expensive to enumerate all substrings.

Intuitively, the smaller size of $\mathcal{W}(s,l)$, the higher performance. Thus we want to find substring sets with smaller sizes. In this section, we propose several methods to select the substring set $\mathcal{W}(s,l)$. As $\mathcal{W}(s,l)=\cup_{i=1}^{\tau+1}\mathcal{W}(s,\mathcal{L}_l^i)$ and we want to use index $\mathcal{L}_l^i$ to do efficient filtering, next we focus on how to generate $\mathcal{W}(s,\mathcal{L}_l^i)$ for $\mathcal{L}_l^i$.

Length-based Method: As segments in $\mathcal{L}_l^i$ have the same length, denoted by l_i, the length-based method selects all substrings of s with length l_i, denoted by $\mathcal{W}_\ell(s,\mathcal{L}_l^i)$. Let $\mathcal{W}_\ell(s,l)=\cup_{i=1}^{\tau+1}\mathcal{W}_\ell(s,\mathcal{L}_l^i)$. The length-based method satisfies completeness, as it selects all substrings with length l_i. The size of $\mathcal{W}_\ell(s,\mathcal{L}_l^i)$ is $|\mathcal{W}_\ell(s,\mathcal{L}_l^i)|$=$|s|-l_i+1$, and the number of selected substrings is $|\mathcal{W}_\ell(s,l)|$=$(\tau+1)(|s|+1)-l$.

Shift-based Method: However the length-based method does not consider the positions of segments. To address this problem, Wang et al. [22] proposed a shift-based method to address the entity identification problem. We can extend their method to support our problem as follows. As segments in $\mathcal{L}_l^i$ have the same length, they have the same start position, denoted by p_i, where $p_1=1$ and p_i=p_1+$\sum_{k=1}^{i-1}l_k$ for $i>1$. The shift-based method selects s's substrings with start positions in $[p_i-\tau, p_i+\tau]$ and with length l_i, denoted by $\mathcal{W}_f(s,\mathcal{L}_l^i)$. Let $\mathcal{W}_f(s,l)=\cup_{i=1}^{\tau+1}\mathcal{W}_f(s,\mathcal{L}_l^i)$. The size of $\mathcal{W}_f(s,\mathcal{L}_l^i)$ is $|\mathcal{W}_f(s,\mathcal{L}_l^i)|$=$2\tau+1$. The number of selected substrings is $|\mathcal{W}_f(s,l)|$=$(\tau+1)(2\tau+1)$.

The basic idea behind the method is as follows. Suppose a substring s_m of s with start position smaller than $p_i-\tau$ or larger than $p_i+\tau$ matches a segment in $\mathcal{L}_l^i$. Consider a string $r\in\mathcal{L}_l^i(s_m)$. We can partition $s(r)$ into three parts: the matching part $s_m(r_m)$, the left part before the matching part $s_l(r_l)$, and the right part after the matching part $s_r(r_r)$. As the start position of r_m is p_i and the start position of s_m is smaller than $p_i-\tau$ or larger than $p_i+\tau$, the length difference between s_l and r_l must be larger than τ. If we align the two strings by matching s_m and r_m (i.e., transforming r_l to s_l, matching r_m with s_m, and transforming r_r to s_r), they will not be similar, thus we can prune substring s_m. Hence the shift-based method satisfies completeness.

However, the shift-based method still involves many unnecessary substrings. For example, consider two strings s_1 = "vankatesh" and s_2 = "avataresha". Suppose $\tau=3$ and "vankatesh" is partitioned into four segments {va, nk, at, esh}. s_2 = "avataresha" contains a substring "at" which matches the third segment in "vank$\underline{\text{at}}$esh", the shift-based method will select it as a substring. However we can prune it and the reason is as follows. Suppose we partition the two strings into three parts based on the matching segment. For instance, we partition "vankatesh" into {"vank" , "at", "esh"}, and "avataresha" into {"av", "at", "aresha"}. Obviously the minimal edit distance (length difference) between the left parts ("vank" and "av") is 2 and the minimal edit distance (length difference) between the right parts ("esh" and "aresha") is 3. Thus if we align the two strings using the matching segment "at", they will not be similar. In this way, we can prune the substring "at".

4.1 Position-aware Substring Selection

Notice that all the segments in $\mathcal{L}_l^i$ have the same length l_i and the same start position p_i. Without loss of generality, we consider a segment $r_m\in\mathcal{L}_l^i$. Moreover, all the strings in

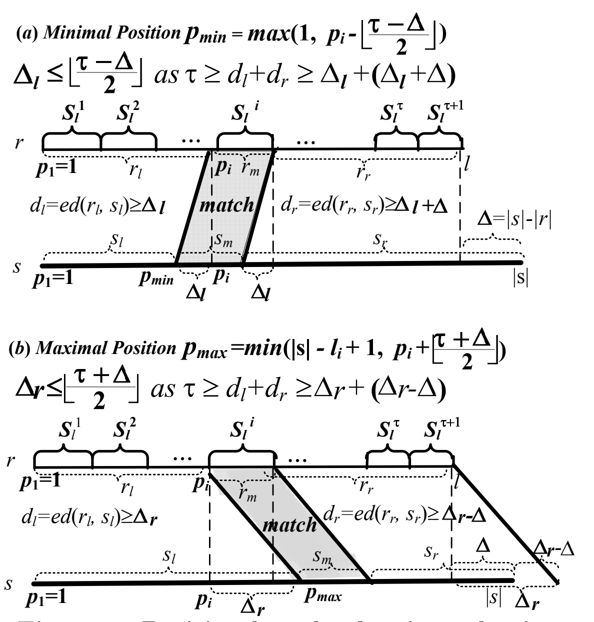

(a) Minimal Position $p_{min} = max(1, p_i - \lfloor\frac{\tau-\Delta}{2}\rfloor)$
$\Delta_l \le \lfloor\frac{\tau-\Delta}{2}\rfloor$ as $\tau \ge d_l+d_r \ge \Delta_l + (\Delta_l+\Delta)$
$d_l=ed(r_l, s_l)\ge\Delta_l$
match
$d_r=ed(r_r, s_r)\ge\Delta_l+\Delta$
$\Delta=|s|-|r|$
(b) Maximal Position $p_{max} = min(|s| - l_i + 1, p_i + \lfloor\frac{\tau+\Delta}{2}\rfloor)$
$\Delta_r \le \lfloor\frac{\tau+\Delta}{2}\rfloor$ as $\tau \ge d_l+d_r \ge \Delta_r + (\Delta_r-\Delta)$
$d_l=ed(r_l, s_l)\ge\Delta_r$
$d_r=ed(r_r, s_r)\ge\Delta_r-\Delta$

Figure 4: Position-based substring selection

inverted list $\mathcal{L}_l^i(r_m)$ have the same length l ($l \le |s|$), and we consider a string r that contains segment r_m. Suppose s has a substring s_m which matches r_m. Next we give the possible start positions of s_m. We still partition $s(r)$ into three parts: the matching part $s_m(r_m)$, the left part $s_l(r_l)$, and the right part $s_r(r_r)$. If we align r and s by matching $r_m = s_m$, that is we transform r to s by first transforming r_l to s_l with $d_l = \text{ED}(r_l, s_l)$ edit operations, then matching r_m with s_m, and finally transforming r_r to s_r with $d_r = \text{ED}(r_r, s_r)$ edit operations, the total transformation distance is $d_l + d_r$. If s is similar to r, $d_l + d_r \le \tau$. Based on this observation, we give s_m's minimal start position (p_{min}) and the maximal start position (p_{max}) as illustrated in Figure 4.

Minimal Start Position: Suppose the start position of s_m, denoted by p, is not larger than p_i. Let $\triangle = |s| - |r|$ and $\triangle_l = p_i - p$. We have $d_l = \text{ED}(r_l, s_l) \ge \triangle_l$ and $d_r = \text{ED}(r_r, s_r) \ge \triangle_l + \triangle$, as illustrated in Figure 4(a). If s is similar to r (or ***any*** string in $\mathcal{L}_l^i(r_m)$), we have

$$\triangle_l + (\triangle_l + \triangle) \le d_l + d_r \le \tau.$$

That is $\triangle_l \le \lfloor\frac{\tau-\triangle}{2}\rfloor$ and $p = p_i - \triangle_l \ge p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor$. Thus $p_{min} \ge p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor$. As $p_{min} \ge 1$, $p_{min} = \max(1, p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor)$.

Maximal Start Position: Suppose the start position of s_m, p, is larger than p_i. Let $\triangle = |s| - |r|$ and $\triangle_r = p - p_i$. We have $d_l = \text{ED}(r_l, s_l) \ge \triangle_r$ and $d_r = \text{ED}(r_r, s_r) \ge |\triangle_r - \triangle|$ as illustrated in Figure 4(b). If $\triangle_r \le \triangle$, $d_r \ge \triangle - \triangle_r$. Thus $\triangle = \triangle_r + (\triangle - \triangle_r) \le d_l + d_r \le \tau$, and in this case, the maximal value of $\triangle_r$ is $\triangle$; otherwise if $\triangle_r > \triangle$, $d_r \ge \triangle_r - \triangle$. If s is similar to r (or ***any*** string in $\mathcal{L}_l^i(r_m)$), we have

$$\triangle_r + (\triangle_r - \triangle) \le d_l + d_r \le \tau.$$

That is $\triangle_r \le \lfloor\frac{\tau+\triangle}{2}\rfloor$, and $p = p_i + \triangle_r \le p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor$. Thus $p_{max} \le p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor$. As the segment length is l_i, based on the boundary, we have $p_{max} \le |s| - l_i + 1$. Thus $p_{max} = \min(|s| - l_i + 1, p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor)$.

For example, consider string r = "`vankatesh`". Suppose $\tau = 3$ and "`vankatesh`" is partitioned into four segments, {`va`, `nk`, `at`, `esh`}. For string s = "`avataresha`", we have $\triangle = |s| - |r| = 1$. $\triangle_l \le \lfloor\frac{\tau-\triangle}{2}\rfloor = 1$ and $\triangle_r \le \lfloor\frac{\tau+\triangle}{2}\rfloor = 2$.

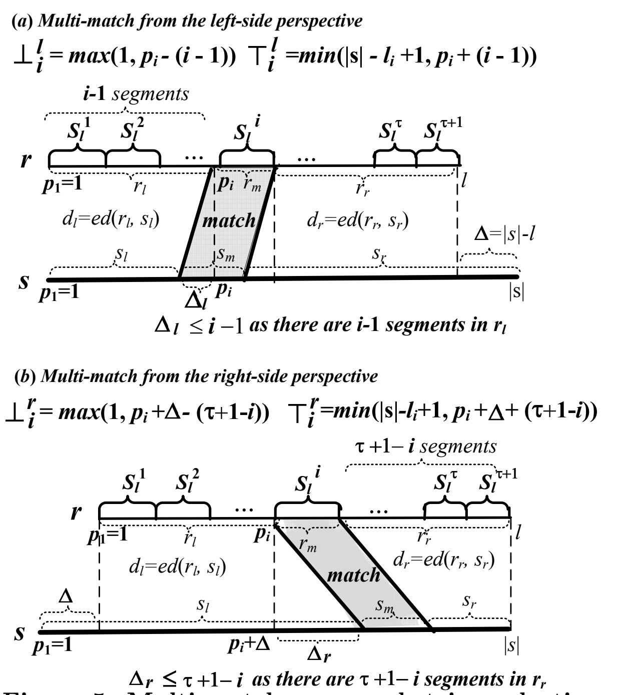

(a) Multi-match from the left-side perspective
$\bot_i^l = max(1, p_i - (i-1))$ $\top_i^l = min(|s| - l_i + 1, p_i + (i-1))$
i-1 segments
$d_l=ed(r_l, s_l)$
match
$d_r=ed(r_r, s_r)$
$\Delta=|s|-l$
$\Delta_l \le i-1$ as there are i-1 segments in r_l
(b) Multi-match from the right-side perspective
$\bot_i^r = max(1, p_i+\Delta-(\tau+1-i))$ $\top_i^r = min(|s|-l_i+1, p_i+\Delta+(\tau+1-i))$
$\tau+1-i$ segments
$\Delta_r \le \tau+1-i$ as there are $\tau+1-i$ segments in r_r

Figure 5: Multi-match-aware substring selection

For the first segment "`va`", $p_1 = 1$. $p_{min} = \max(1, p_1 - \lfloor\frac{\tau-\triangle}{2}\rfloor) = 1$ and $p_{max} = 1 + \lfloor\frac{\tau+\triangle}{2}\rfloor = 3$. Thus we only need to enumerate the following substrings "`av`", "`va`", "`at`" for the first segment. Similarly, we need to enumerate substrings "`va`", "`at`", "`ta`", "`ar`" for the second segment, "`ta`", "`ar`", "`re`", "`es`" for the third segment, and "`res`", "`esh`", "`sha`" for the fourth segment. We see that the position-based method can reduce many unnecessary substrings over the shift-based method (reducing the number from 28 to 14).

For $\mathcal{L}_l^i$, the position-aware method selects substrings with start positions in $[p_{min}, p_{max}]$ and with length l_i, denoted by $\mathcal{W}_p(s, \mathcal{L}_l^i)$. Let $\mathcal{W}_p(s, l) = \cup_{i=1}^{\tau+1}\mathcal{W}_p(s, \mathcal{L}_l^i)$. The size of $\mathcal{W}_p(s, \mathcal{L}_l^i)$ is $|\mathcal{W}_p(s, \mathcal{L}_l^i)| = \tau + 1$ and the number of selected substrings is $|\mathcal{W}_p(s, l)| = (\tau+1)^2$. The position-aware method satisfies completeness as formalized in Theorem 1.

THEOREM 1.*The position-aware substring selection method satisfies completeness.*

4.2 Multi-match-aware Substring Selection

We have an observation that string s may have multiple substrings that match some segments of string r. In this case we can discard some of these substrings. For example, consider r = "`vankatesh`" with four segments, {`va`, `nk`, `at`, `esh`}. s = "`avataresha`" has three substrings `va`, `at`, `esh` matching the segments of r. We can discard some of these substrings. To this end, we propose a multi-match-aware substring selection method.

Consider $\mathcal{L}_l^i$. Suppose string s has a substring s_m that matches a segment in $\mathcal{L}_l^i$. If we know that s *must have* a substring after s_m which will match a segment in $\mathcal{L}_l^j (j > i)$, we can discard substring s_m. For example, s = "`avataresha`" has a substring "`va`" matching a segment in r = "`vankatesh`". Consider the three parts $r_m = s_m$ = "`va`", $r_l = \phi$ and s_l = "`a`", and r_r = "`nkatesh`" and s_r = "`taresha`". As $d_l \ge 1$, if s and r are similar, $d_r \le \tau - d_l \le \tau - 1 = 2$. As there are still 3 segments in r_r, thus s_r must have a substring matching a segment in r_r based on the pigeon-hole principle. Thus we can discard the substring "`va`" and use the next substring to find similar pairs. Next we generalize our idea.

Suppose s has a substring s_m with start position p matching a segment $r_m \in \mathcal{L}_l^i$. We still consider the three parts of the two strings: s_l, s_m, s_r and r_l, r_m, r_r as illustrated in Figure 5. Let $\triangle_l = |p_i - p|$. $d_l = \text{ED}(r_l, s_l) \geq \triangle_l$. As there are $i-1$ segments in s_l, if each segment only has 1 error when transforming r_l to s_l, we have $\triangle_l \leq i-1$. If $\triangle_l \geq i$, $d_l = \text{ED}(r_l, s_l) \geq \triangle_l \geq i$, $d_r = \text{ED}(r_r, s_r) \leq \tau - d_l \leq \tau - i$ (if s is similar to r). As r_r contains $\tau + 1 - i$ segments, s_r must contain a substring matching a segment in r_r based on the pigeon-hole principle, which can be proved similar to Lemma 1. In this way, we can discard s_m, since for ***any*** string $r \in \mathcal{L}_l^i(r_m)$, s must have a substring that matches a segment in the right part r_r, and thus we can identify strings similar to s using the next matching segment. In summary, if $\triangle_l = |p - p_i| \leq i-1$, we keep the substring with start position p for $\mathcal{L}_l^i$. That is the minimal start position is $\perp_i^l = \max\big(1, p_i - (i-1)\big)$ and the maximal start position is $\top_i^l = \min\big(|s| - l_i + 1, p_i + (i-1)\big)$.

For example, consider string r="`vankatesh`" with four segments, {`va`, `nk`, `at`, `esh`}, and string s="`avataresha`". For the first segment, we have $\perp_i^l$=1-0=1 and $\top_i^l$=1+0=1. Thus the selected substring is only "`av`" for the first segment. For the second segment, we have $\perp_i^l$=3-1=2 and $\top_i^l$=3+1=4. Thus the selected substrings are "`va`", "`at`", and "`ta`" for the second segment. Similarly for the third segment, we have $\perp_i^l$=5-2=3 and $\top_i^l$=5+2=7, and for the fourth segment, we have $\perp_i^l$=7-3=4 and $\top_i^l$=7+3=10.

The above observation is made from the left-side perspective. Similarly, we can use the same idea from the right-side perspective. As there are $\tau+1-i$ segments on the right part r_r, there are at most $\tau+1-i$ edit errors on r_r. If we transform r to s from the right-side perspective, position p_i on r should be aligned with position $p_i + \triangle$ on s as shown in Figure 5(b). Suppose the position p on s matching position p_i on r. Let $\triangle_r = |p - (p_i + \triangle)|$. We have $d_r = \text{ED}(s_r, r_r) \geq \triangle_r$. As there are $\tau+1-i$ segments on the right part r_r, we have $\triangle_r \leq \tau + 1 - i$. Thus the minimal start position for $\mathcal{L}_l^i$ is $\perp_i^r = \max\big(1, p_i + \triangle - (\tau + 1 - i)\big)$ and the maximal start position is $\top_i^r = \min\big(|s| - l_i + 1, p_i + \triangle + (\tau + 1 - i)\big)$.

Consider the above example. Suppose $\tau = 3$ and $\triangle = 1$. For the fourth segment, we have $\perp_i^r = 7+1-(3+1-4) = 8$ and $\top_i^r = 7+1+(3+1-4) = 8$. Thus the selected substring is only "`sha`" for the fourth segment. Similarly for the third segment, we have $\perp_i^r = 5$ and $\top_i^r = 7$. Thus the selected substrings are "`ar`", "`re`", and "`es`" for the third segment.

More interestingly, we can use the two techniques simultaneously. That is for $\mathcal{L}_l^i$, we only select the substrings with the start positions between $\perp_i = \max(\perp_i^l, \perp_i^r)$ and $\top_i = \min(\top_i^l, \top_i^r)$ and with length l_i, denoted by $\mathcal{W}_m(s, \mathcal{L}_l^i)$. Let $\mathcal{W}_m(s,l)$=$\cup_{i=1}^{\tau+1}\mathcal{W}_m(s, \mathcal{L}_l^i)$. The number of selected substrings is $|\mathcal{W}_m(s,l)|$=$\lfloor\frac{\tau^2-\triangle^2}{2}\rfloor$+$\tau$+1 as stated in Lemma 2.

LEMMA 2. $|\mathcal{W}_m(s,l)| = \lfloor\frac{\tau^2-\triangle^2}{2}\rfloor + \tau + 1$.

Moreover we prove that the multi-match-aware selection method satisfies completeness as stated in Theorem 2.

THEOREM 2. *The multi-match-aware substring selection method satisfies completeness.*

Consider the above example. For the first segment, we have $\perp_i = 1 - 0 = 1$ and $\top_i = 1 + 0 = 1$. We select "`av`" for the first segment. For the second segment, we have $\perp_i = 3 - 1 = 2$ and $\top_i = 3 + 1 = 4$. We select substrings "`va`", "`at`", and "`ta`" for the second segment. For the third segment, we have $\perp_i = 5 + 1 - (3 + 1 - 3) = 5$ and $\top_i = 5 + 1 + (3 + 1 - 3) = 7$. We select substrings "`ar`", "`re`", and "`es`" for the third segment. For the fourth segment, we have $\perp_i = 7 + 1 - (3 + 1 - 4) = 8$ and $\top_i = 7 + 1 + (3 + 1 - 4) = 8$. Thus we select the substring "`sha`" for the fourth segment. The multi-match-aware method only selects 8 substrings.

4.3 Comparison of Selection Methods

We compare the selected substring sets of different methods. Let $\mathcal{W}_\ell(s,l), \mathcal{W}_f(s,l), \mathcal{W}_p(s,l), \mathcal{W}_m(s,l)$ respectively denote the sets of selected substrings that use the length-based selection method, the shift-based selection method, the position-aware selection method, and the multi-match-aware selection method. Based on the size analysis of each set, we have $|\mathcal{W}_m(s,l)| \leq |\mathcal{W}_p(s,l)| \leq |\mathcal{W}_f(s,l)| \leq |\mathcal{W}_\ell(s,l)|$. Next we prove $\mathcal{W}_m(s,l) \subseteq \mathcal{W}_p(s,l) \subseteq \mathcal{W}_f(s,l) \subseteq \mathcal{W}_\ell(s,l)$ as formalized in Lemma 3.

LEMMA 3. *For any string s and a length l, we have $\mathcal{W}_m(s,l) \subseteq \mathcal{W}_p(s,l) \subseteq \mathcal{W}_f(s,l) \subseteq \mathcal{W}_\ell(s,l)$.*

Moreover, we can prove that $\mathcal{W}_m(s,l)$ has the minimum size among all substring sets generated by the methods that satisfy completeness as formalized in Theorem 3.

THEOREM 3. *The substring set $\mathcal{W}_m(s,l)$ generated by the multi-match-aware selection method has the minimum size among all the substring sets generated by the substring selection methods that satisfy completeness.*

Theorem 3 proves that the substring set $\mathcal{W}_m(s,l)$ has the minimum size. Next we introduce another concept to show the superiority of our multi-match-aware selection method.

DEFINITION 3 (MINIMALITY). *A substring set $\mathcal{W}(s,l)$ generated by a method with the completeness property satisfies minimality, if for any substring set $\mathcal{W}'(s,l)$ generated by a method with the completeness property, $\mathcal{W}(s,l) \subseteq \mathcal{W}'(s,l)$.*

Next we prove that if $l \geq 2(\tau+1)$ and $|s| \geq l$, the substring set $\mathcal{W}_m(s,l)$ generated by our multi-match-aware selection method satisfies minimality as stated in Theorem 4. The condition $l \geq 2(\tau + 1)$ makes sense where each segment is needed to have at least two characters. For example, if $10 \leq l < 12$, we can tolerate $\tau = 4$ edit operations. If $12 \leq l < 14$, we can tolerate $\tau = 5$ edit operations.

THEOREM 4. *If $l \geq 2(\tau+1)$ and $|s| \geq l$, $\mathcal{W}_m(s,l)$ satisfies minimality.*

4.4 Substring-selection Algorithm

Based on above discussion, we improve SUBSTRINGSELECTION algorithm by avoiding unnecessary substrings. For $\mathcal{L}_l^i$, we use the multi-match-aware selection method to select substrings, and the selection complexity is $\mathcal{O}(\tau)$. Figure 6 gives the pseudo-code of the selection algorithm.

For example, consider the strings in Table 1. We create inverted indices as illustrated in Figure 1. Consider string s_1 = "`vankatesh`" with four segments, we build four inverted lists for its segments {`va`, `nk`, `at`, `esh`}. Then for s_2 = "`avataresha`". We use multi-match-aware selection method to select its substrings. Here we only select 8 substrings for s_2 and use the 8 substrings to find similar strings of s_2 from the inverted indices. Similarly, we can select substrings and find similar string pairs for other strings.

```
Algorithm 2: SubstringSelection($s, \mathcal{L}_l^i$)
  Input: $s$: A string; $\mathcal{L}_l^i$: Inverted index
  Output: $\mathcal{W}(s, \mathcal{L}_l^i)$: Selected substrings
1 begin
2     for $p \in [\perp_i, \top_i]$ do
3         Add the substring of $s$ with start position $p$ and
          with length $l_i$ ($s[p, l_i]$) into $\mathcal{W}(s, \mathcal{L}_l^i)$;
4 end
```

Figure 6: SubstringSelection algorithm

5. IMPROVING THE VERIFICATION

In our framework, for string s and inverted index $\mathcal{L}_l^i$, we generate a set of its substrings $\mathcal{W}(s, \mathcal{L}_l^i)$. For each substring $w \in \mathcal{W}(s, \mathcal{L}_l^i)$, we need to check whether it appears in $\mathcal{L}_l^i$. If $w \in \mathcal{L}_l^i$, for each string $r \in \mathcal{L}_l^i(w)$, $\langle r, s\rangle$ is a candidate pair and we need to verify the candidate pair. In this section we propose effective techniques to do efficient verification.

5.1 Length-aware Verification

Given a candidate pair $\langle r, s\rangle$, a straightforward method to verify the pair is to use a dynamic-programming algorithm to compute their real edit distance. If the edit distance is not larger than τ, the pair is an answer. We can use a matrix M with $|r|+1$ rows and $|s|+1$ columns to compute their edit distance, in which $M(0, j) = j$ for $0 \le j \le |s|$, and for $1 \le i \le |r|$ and $0 \le j \le |s|$,

$$M(i,j) = \min\big(M(i-1,j)+1, M(i,j-1)+1, M(i-1,j-1)+\delta\big)$$

where $\delta = 0$ if the i-th character of r is the same as the j-th character of s; otherwise $\delta = 1$. The time complexity of the dynamic-programming algorithm is $\mathcal{O}(|r| * |s|)$.

Actually, we do not need to compute their real edit distance and only need to check whether their edit distance is not larger than τ. An improvement based on length pruning [20] is proposed which only computes the values $M(i, j)$ for $|i - j| \le \tau$, as shown in the shaded cells of Figure 7(a). The basic idea is that if $|i - j| > \tau$, $M(i, j) > \tau$, and we do not need to compute such values. This method improves the time complexity $\mathcal{V}(s, r)$ to $\mathcal{O}\big((2*\tau+1)*\min(|r|, |s|)\big)$. Next, we propose a technique to further improve the performance by considering the length difference between r and s.

We first use an example to illustrate our idea. Consider string r = "`kaushuk chadhui`" and string s = "`caushik chakrabar`". Suppose $\tau = 3$. Existing methods need to compute all the shaded values in Figure 7(a). We have an observation that we do not need to compute $M(2, 1)$, which is the edit distance between "`ka`" and "`c`". This is because if there is a transformation from r to s by first transforming "`ka`" to "`c`" with at least 1 edit operation (length difference) and then transforming "`ushuk chadhui`" to "`aushik chakrabar`" with at least 3 edit operations (length difference), the transformation distance is at least 4 which is larger than $\tau = 3$. In other words, even if we do not compute $M(2, 1)$, we know that there is no transformation including $M(2, 1)$ (the transformation from "`ka`" to "`c`") whose edit distance is not larger than τ. Actually we only need to compute the highlighted values as illustrated in Figure 7(b).

To address this problem, we propose a length-aware verification method. Without loss of generality, let $|s| \ge |r|$ and $\triangle = |s| - |r| \le \tau$ (otherwise their edit distance must be larger than τ). We call a transformation from r to s including $M(i, j)$, if the transformation first transforms the

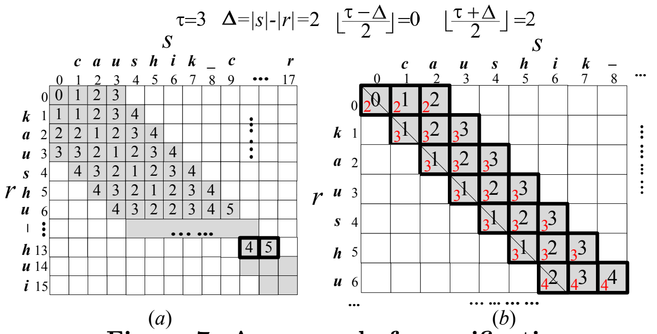


Figure 7: An example for verification

first i characters of r to the first j characters of s with d_1 edit operations and then transforming the other characters in r to the other characters in s with d_2 edit operations. Based on length difference, we have $d_1 \ge |i - j|$ and $d_2 \ge |(|s| - j) - (|r| - i)| = |\triangle + (i - j)|$. If $d_1 + d_2 > \tau$, we do not need to compute $M(i, j)$, since the distance of any transformation including $M(i, j)$ is larger than τ. To check whether $d_1 + d_2 > \tau$, we consider the following cases.

(1) If $i \ge j$, we have $d_1 + d_2 \ge i - j + \triangle + i - j$. If $i-j+\triangle+i-j > \tau$, that is $j < i - \frac{\tau-\triangle}{2}$, we do not need to compute $M(i, j)$. In other words, we only need to compute $M(i, j)$ with $j \ge i - \frac{\tau-\triangle}{2}$.

(2) If $i < j$, $d_1 = j - i$. If $j - i \le \triangle$, $d_1 + d_2 \ge j - i + \triangle - (j - i) = \triangle$. As $\triangle \le \tau$, there is no position constraint. We need to compute $M(i, j)$; otherwise if $j - i > \triangle$, we have $d_1 + d_2 \ge j - i + j - i - \triangle$. If $j - i + j - i - \triangle > \tau$, that is $j > i + \frac{\tau+\triangle}{2}$, we do not need to compute $M(i, j)$. In other words, we only need to compute $M(i, j)$ with $j \le i + \frac{\tau+\triangle}{2}$.

Based on this observation, for each row $M(i, *)$, we only compute $M(i, j)$ for $i - \lfloor\frac{\tau-\triangle}{2}\rfloor \le j \le i + \lfloor\frac{\tau+\triangle}{2}\rfloor$. For example, in Figure 8, we only need to compute the values in black circles. Thus we can improve the time complexity $\mathcal{V}(s, r)$ from $\mathcal{O}\big((2\tau+1)*\min(|r|, |s|)\big)$ to $\mathcal{O}\big((\tau+1)*\min(|r|, |s|)\big)$.

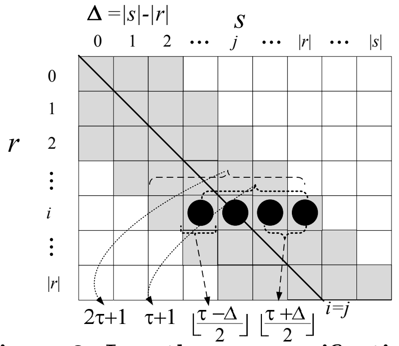


Figure 8: Length-aware verification

Early Termination: We can further improve the performance by doing an early termination. Consider the values in row $M(i, *)$. A straightforward early-termination method is to check each value in $M(i, *)$, and if each value is larger than τ, we can do an early termination. This is because the values in the following rows $M(k > i, *)$ must be larger than τ based on the dynamic-programming algorithm. This pruning technique is called *prefix pruning*. For example in Figure 7(a), if $\tau = 3$, after we have computed $M(13, *)$, we can do an early termination as all the values in $M(13, *)$ are larger than τ. But in our method, after we have computed the values in $M(6, *)$, we can conclude that the edit distance between the two strings is at least 4 (larger than $\tau = 3$). Thus we do not need to compute $M(i > 6, *)$ and can terminate the computation as shown in Figure 7(b). To this end, we propose a novel early-termination method.

For ease of presentation, we first introduce several notations. Given a string s, let $s[i]$ denote the i-th character and $s[i : j]$ denote the substring of s from the i-th character to the j-th character. Notice that $M(i, j)$ denotes the edit distance between $r[1 : i]$ and $s[1 : j]$. We can estimate the lower bound of the edit distance between $r[i : |r|]$ and $s[j : |s|]$ using their length difference $\big|(|s|-j)-(|r|-i)\big|$. We use $E(i, j) = M(i, j) + \big|(|s| - j) - (|r| - i)\big|$ to estimate the edit distance between s and r, which is called *expected edit distance* of s and r with respect to $M(i, j)$. If each expected edit distance for $M(i, j)$ in $M(i, *)$ is larger than τ, the edit distance between r and s must be larger than τ, thus we can do an early termination. To achieve our goal, for each value $M(i, j)$, we maintain the expected minimal edit distance $E(i, j)$. If each value in $E(i, *)$ is larger than τ, we can do an early termination as formalized in Lemma 4.

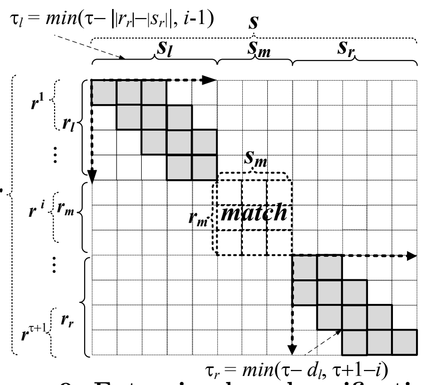
Lemma 4. *Given strings s and r, if each value in $E(i, *)$ is larger than τ, the edit distance of r and s is larger than τ.*

For example, in Figure 7(b), we show the expected edit distances in the left-bottom corner of each cell. When we have computed $M(6, *)$ and $E(6, *)$, all values in $E(6, *)$ are larger than 3, thus we can do an early termination. In this way, we can avoid many unnecessary computations. Note that our proposed verification techniques can be applied to any other algorithms to verify a candidate pair in terms of edit distance (e.g., ED-Join and NGPP).

5.2 Extension-based Verification

Consider a selected substring w of string s. If w appears in the inverted index $\mathcal{L}_l^i$, for each string r in the inverted list $\mathcal{L}_l^i(w)$, we need to verify the pair $\langle s, r\rangle$. As s and r share a common segment w, we can use the shared segment to efficiently verify the pair. To achieve our goal, we propose an extension-based verification algorithm.

As r and s share a common segment w, we partition them into three parts based on the common segment. We partition r into three parts, the left part r_l, the matching part $r_m = w$, and the right part r_r. Similarly, we get three parts for string s: s_l, $s_m = w$, and s_r. Here we align s and r based on the matching substring r_m and s_m, and we only need to verify whether r and s are similar in this alignment. Thus we first compute the edit distance $d_l = \text{ED}(r_l, s_l)$ between r_l and s_l using the above-mentioned method. If d_l is larger than τ, we terminate the computation; otherwise, we compute the edit distance $d_r = \text{ED}(s_r, r_r)$ between s_r and r_r. If $d_l + d_r$ is larger than τ, we discard the pair; otherwise we take it as an answer. Note that this method will not involve false negatives. This is because based on Lemma 1, if s and r are similar, s must have a substring that matches a segment of r. In addition, based on dynamic-programming algorithm, there must exist a transformation by aligning r_m with s_m and $\text{ED}(s, r) = d_l + d_r$. As our method selects all possible substrings and considers all such common segments, our method will not miss any results. On the other hand, as we find the answers with $d_l + d_r \le \tau$ and $\text{ED}(s, r) \le d_l + d_r \le \tau$, our method will not involve false positives. To guarantee correctness of our extension-based method. We first give a formal definition of correctness.



$\tau_l = min(\tau - ||r_r|-|s_r||, i\text{-}1)$
s
s_l
s_m
s_r
r^1
r_l
s_m
r
r^i
r_m
r_m
match
$r^{\tau+1}$
r_r
$\tau_r = min(\tau - d_l, \tau+1-i)$

Figure 9: Extension-based verification

Definition 4 (Correctness). *Given a candidate pair $\langle s, r\rangle$, a verification algorithm is correct, if it satisfies (1) If $\langle s, r\rangle$ passes the algorithm, $\langle s, r\rangle$ must be a similar pair; and (2) If $\langle s, r\rangle$ is a similar pair, it must pass the algorithm.*

We then prove that our extension-based verification method satisfies correctness as stated in Theorem 5.

Theorem 5. *Our extension-based verification method satisfies correctness.*

Actually, we can further improve the verification algorithm. For the left parts, we can give a tighter threshold $\tau_l \le \tau$. The basic idea is as follows. As the minimal edit distance between the two right segments r_r and s_r is $\big||r_r| - |s_r|\big|$. Thus we can set $\tau_l = \tau - \big||r_r| - |s_r|\big|$. If the edit distance between r_l and s_l is larger than threshold τ_l, we can terminate the verification; otherwise we continue to compute $d_r = \text{ED}(r_r, s_r)$. Similarly for the right parts, we can also give a tighter threshold $\tau_r \le \tau$. As d_l has been computed, we can set $\tau_r = \tau - d_l$ as a threshold to verify whether r_r and s_r are similar. If d_r is larger than threshold τ_r, we can terminate the verification.

For example, if we want to verify s_5 = "`kaushuk chadhui`" and s_6 = "`caushik chakrabar`". s_5 and s_6 share a segment "`_cha`". We have s_{5_l} = "`kaushuk`" and s_{6_l} = "`caushik`", and s_{5_r} = "`dhui`" and s_{6_r} = "`krabar`". Suppose $\tau = 3$. As $\Big||s_{5_r}| - |s_{6_r}|\Big| = 2$, $\tau_l = \tau - 2 = 1$. We only need to verify whether the edit distance between s_{5_l} and s_{6_l} is not larger than $\tau_l = 1$. After we have computed $M(6, *)$, we can do an early termination as each value in $E(6, *)$ is larger than 1, as shown in Figure 7. Note that as $\tau_l = 1$ and $|s_{5_l}| - |s_{6_l}| = 0$, $\bot_i = \top_i = 0$. Thus we only need to compute $M(i, i)$.

We discuss how to deduce a tighter bound for τ_l and τ_r. Consider the i-th segment. If $d_l \ge i$, we can terminate the verification based on the multi-match-aware method. Thus we have $\tau_l = i - 1$. Combining with the above pruning condition, we have $\tau_l = \min(\tau - \big||r_r| - |s_r|\big|, i - 1)$. As $\big||r_r| - |s_r|\big| = \big|(|r| - p_i - l_i) - (|s| - p - l_i)\big| = |p - p_i - \triangle| \le \tau + 1 - i$ (based on the multi-match-aware method), $\tau - \big||r_r| - |s_r|\big| \ge i - 1$. We set $\tau_l = i - 1$. Similarly we have $\tau_r = \min(\tau - d_l, \tau + 1 - i)$. As $d_l \le \tau_l \le i - 1$, $\tau - d_l \ge \tau - (i - 1)$. Thus we set $\tau_r = \tau + 1 - i$.

5.3 Sharing Computations

Given a selected substring w, there may be large numbers of strings in $\mathcal{L}_l^i(w)$ similar to string s. When computing the edit distance of the left parts s_l and r_l (and that of the right parts s_r and r_r), we can share the computations if they have common prefixes. Next we discuss how to share computations. As the strings in $\mathcal{L}_l^i(w)$ are sorted in alphabetical

Algorithm 3: VERIFICATION($s, \mathcal{L}_l^i(w), \tau$)

```
Input: $s$: A string; $\mathcal{L}_l^i(w)$: Inverted list; $\tau$: Threshold
Output: $\mathcal{R} = \{(s \in \mathcal{S}, r \in \mathcal{S}) \mid \text{ED}(s,r) \le \tau\}$
1 begin
2     $\tau_l = i - 1$;
3     $\tau_r = \tau + 1 - i$;
4     for $r \in \mathcal{L}_l^i(w)$ do
5         $d_l = $ VERIFYSTRINGPAIR($s_l, r_l, \tau_l$);
6         if $d_l \le \tau_l$ then
7             $d_r = $ VERIFYSTRINGPAIR($s_r, r_r, \tau_r$);
8             if $d_r \le \tau_r$ then $\mathcal{R} \leftarrow \langle r, s \rangle$;
9 end
```

Function VERIFYSTRINGPAIR(s, r, τ')

```
Input: $s$: A string; $r$: A string; $\tau'$: A threshold
Output: $d = \min(\tau' + 1, \text{ED}(s,r))$
1 begin
2     Using the length-aware verification with the threshold $\tau'$
      and sharing the computations on common prefixes;
3     if Early Termination then $d = \tau' + 1$;
4     else $d = \text{ED}(s,r)$;
5 end
```

Figure 10: Verification algorithm

order, we visit strings in $\mathcal{L}_l^i(w)$ in order. Suppose the first string is r_1 and its three parts are $r_{1_l}, r_{1_m}, r_{1_r}$. We compute the edit distance between r_{1_l} and s_l using the dynamic-programming algorithm. We store the matrix for r_{1_l} and s_l. For the next string r_2 with left part r_{2_l}, we use the stored matrix to compute the edit distance between r_{2_l} and s_l. We first compute the longest common prefix between r_{2_l} and r_{1_l}, denoted by c. When computing the edit distance between s_l and r_{2_l}, we use the stored matrix on s_l and c which has already been computed for s_l and r_{1_l}. Then for the characters after c in r_{2_l}, we continue the computation using the kept matrix. Thus we avoid many unnecessary computations. Notice that we do not need to maintain multiple matrixes and only keep a single matrix for the current string. We use the same idea on the right parts(s_r, r_r).

5.4 Verification Algorithm

Based on our proposed techniques, we improve the VERIFICATION algorithm. Consider a string s, a selected substring w, and an inverted list $\mathcal{L}_l^i(w)$. For $r \in \mathcal{L}_l^i(w)$, we use the extension-based method to verify the candidate pair $\langle s, r\rangle$ as follows. We first compute $\tau_l = i-1$ and $\tau_r = \tau+1-i$. Then for each $r \in \mathcal{L}_l^i(w)$, we compute the edit distance (d_l) between r_l and s_l using the tighter bound τ_l. If $d_l > \tau_l$, we terminate the verification; otherwise we verify whether s_r and r_r are similar with threshold τ_r. When computing the edit distance between s_l and r_l(s_r and r_r), we use the length-aware verification and share the computations on common prefixes. Figure 10 illustrates the pseudo-code.

5.5 Correctness and Completeness

We prove correctness and completeness of our algorithm as formalized in Theorem 6.

THEOREM 6. *Our algorithm satisfies (1) completeness: Given any similar pair $\langle s, r\rangle$, our algorithm must find it as an answer; and (2) correctness: A pair $\langle s, r\rangle$ found in our algorithm must be a similar pair.*

6. EXPERIMENTAL STUDY

We have implemented our method and conducted an extensive set of experimental studies on three real datasets: DBLP Author[†], DBLP Author+Title, and AOL Query Log[‡]. DBLP Author is a dataset with short strings, DBLP Author+Title is a dataset with long strings, and the Query Log is a set of query logs. Table 2 shows the detailed information of the datasets. Note that the Author+Title dataset is the same as that used in ED-JOIN and the Author dataset is the same as that used in TRIE-JOIN. Figure 11 shows the string length distributions of different datasets.

Table 2: Datasets

Datasets	Cardinality	Avg Len	Max Len	Min Len
Author	612781	14.826	46	6
Query Log	464189	44.75	522	30
Author+Title	863073	105.82	886	21

We compared our algorithms with state-of-the-art methods, ED-JOIN [23] and TRIE-JOIN [20]. As ED-JOIN and TRIE-JOIN outperform other methods, e.g., Part-Enum [2] and All-Pairs-Ed [3] (also experimentally proved in [23, 20]), in the paper we only compared our method with the two best studies. We downloaded their binary codes from their homepages, ED-JOIN [§] and TRIE-JOIN [¶].

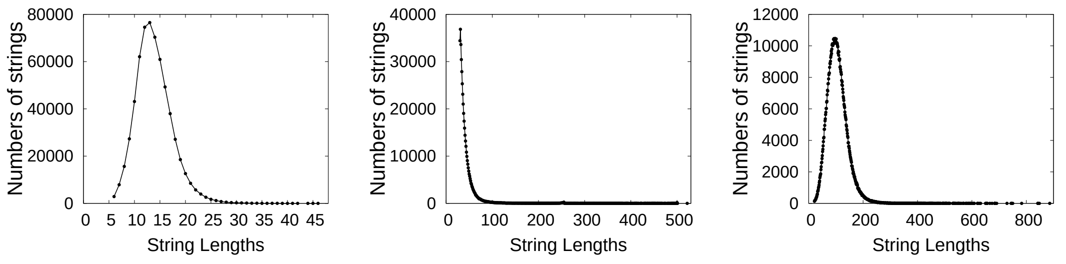
All the algorithms were implemented in C++ and compiled using GCC 4.2.4 with -O3 flag. All the experiments were run on a Ubuntu machine with an Intel Core 2 Quad X5450 3.00GHz processor and 4 GB memory.

6.1 Evaluating Substring Selection

In this section, we evaluate substring selection techniques. We implemented the following four methods. (1) The length-based selection method, denoted by Length, which selects the substrings with the same lengths as the segments. (2) The shift-based method, denoted by Shift, which selects the substring by shifting $[-\tau, \tau]$ positions as discussed in Section 4. (3) Our position-aware selection method, denoted by Position. (4) Our multi-match-aware selection method, denoted by Multi-match. We first compared the total number of selected substrings. Figure 12 shows the results.

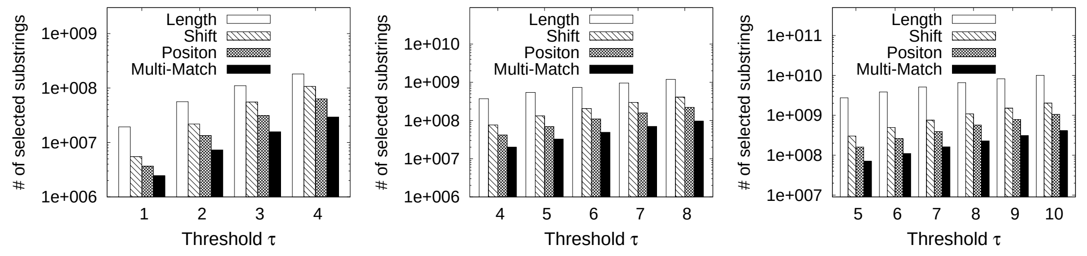
We can see that the Length-based method selected large numbers of substrings. The number of selected substring of the Position-based method was about a tenth to a fourth of that of the Length-based method and a half of the Shift-based method. The Multi-match-based method further reduced the number of selected substrings to about a half of that of the Position-based method. For example, on Author dataset, for $\tau = 1$, the Length-based method selected 19 million substrings, the Shift-based method selected 5.5 million substrings, the Position-based method reduced the number to 3.7 million, and the Multi-match-based method further deceased the number to 2.4 million. Based on our analysis in Section 4, for strings with l, the length-based method selected $(\tau+1)(|s|+1)-l$ substrings, the shift-based method selected $(\tau+1)(2\tau+1)$ substrings, the position-based method selected $(\tau+1)^2$ substrings, and the multi-match-aware method selected $\lfloor \frac{\tau^2-\triangle^2}{2} \rfloor + \tau + 1$ substrings. If $|s|$=l=15 and $\tau = 1$, the number of selected substrings of the four methods are respectively 17, 6, 4, and 2. Obviously the experimental results consisted with our theoretical analysis.

[†]http://www.informatik.uni-trier.de/~ley/db
[‡]http://www.gregsadetsky.com/aol-data/
[§]http://www.cse.unsw.edu.au/~weiw/project/simjoin.html
[¶]http://dbgroup.cs.tsinghua.edu.cn/wangjn/



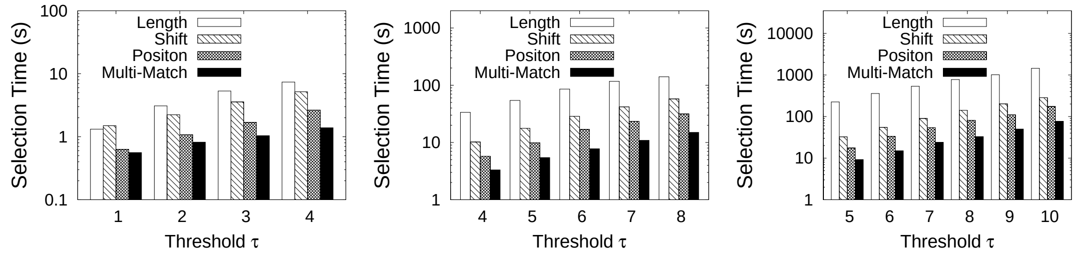
(a) Author (Avg Len = 15) (b) Query Log (Avg Len = 45) (c) Author+Title (Avg Len = 105)

Figure 11: String length distribution



(a) Author (Avg Len = 15) (b) Query Log (Avg Len = 45) (c) Author+Title (Avg Len = 105)

Figure 12: Numbers of selected substrings



(a) Author (Avg Len = 15) (b) Query Log (Avg Len = 45) (c) Author+Title (Avg Len = 105)

Figure 13: Elapsed time for generating substrings

We also compared the elapsed time to generate substrings. Figure 13 shows the results. We see that the Multi-match-based method outperformed the Position-based method which in turns was better than the Shift-based method and the Length-based method. This is because the elapsed time depended on the number of selected substrings and the Multi-match-based selected the smallest number of substrings.

6.2 Evaluating Verification

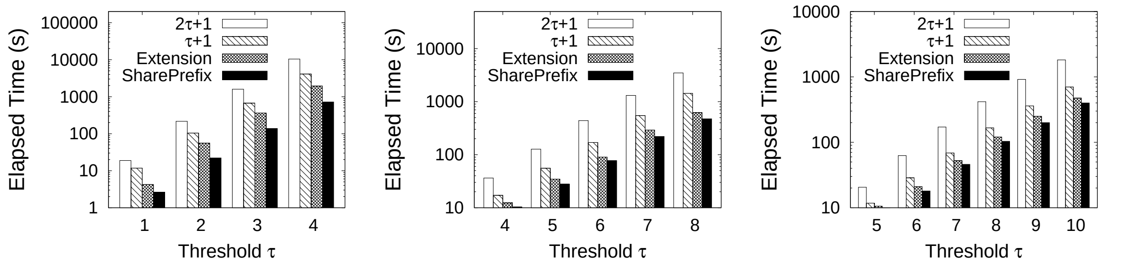
In this section, we evaluate our verification techniques. We implemented four methods. (1) The naive method, denoted by $2\tau+1$, which computed $2\tau+1$ values in each row and used the naive early-termination technique (if all values in a row are larger than τ, we terminate). (2) Our length-aware method, denoted by $\tau+1$, which computed $\tau+1$ values in each row and used the expected edit distance to do early termination. (3) Our extension-based method, denoted by Extension, which used the extension-based framework. It also computed $\tau+1$ rows and used the expected edit distance to do early termination. (4) We used the extension-based method with sharing computations on common prefixes, denoted by SharePrefix. Figure 14 shows the results.

We see that the naive method had the worst performance, as it needed to compute many unnecessary values in the matrix. Our length-aware method was 2-5 times faster than the naive method. This is because our length-aware method can decrease the complexity from $2\tau+1$ to $\tau+1$ and used expected edit distances to do early termination. The extension-based method achieved higher performance and was 2-4 times faster than the length-aware method. The reason is that the extension-based method can avoid the duplicated computations on the common segments and it also used a tighter bound to verify the left parts and the right parts. The SharePrefix method achieved the best performance, as it can avoid many unnecessary computations for strings with common prefixes. For example, on the Author dataset, for $\tau = 4$ the naive method took 10, 000 seconds, the length-aware method decreased the time to 4000 seconds, the extension-based method reduced it to 2000 seconds, and the SharePrefix method further improved the time to about 700 seconds. On the Query Log dataset, for $\tau = 8$, the elapsed time of the four methods were respectively 3500 seconds, 1500 seconds, 600 seconds, and 450 seconds.

6.3 Comparison with Existing Methods

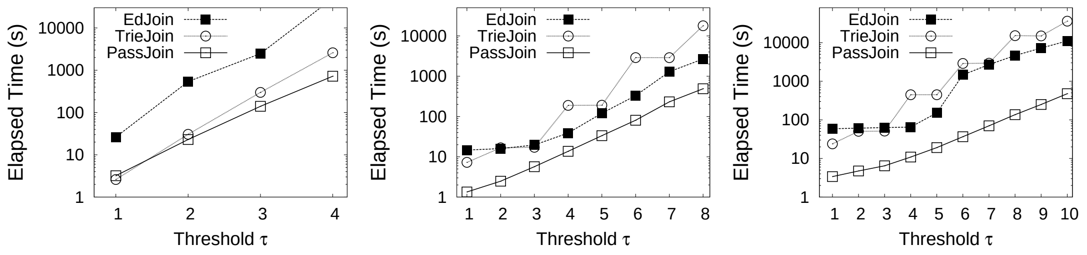
In this section, we compare our method with state-of-the-art methods ED-JOIN [23] and TRIE-JOIN [20]. As TRIE-JOIN had multiple algorithms, we reported the best results. For ED-JOIN, we tuned its parameter q and reported the best results. As TRIE-JOIN was efficient for short strings, we downloaded the same dataset from TRIE-JOIN homepage (i.e., Author with short strings) and used it to compare with TRIE-JOIN. As ED-JOIN was efficient for long strings, we downloaded the same dataset from ED-JOIN homepage (i.e., Author+Title with long strings) and used it to compare with ED-JOIN. Figure 15 shows the results, where the elapsed time included the indexing time and the join time.

On the Author dataset with short strings, TRIE-JOIN outperformed ED-JOIN, and our method was much better than



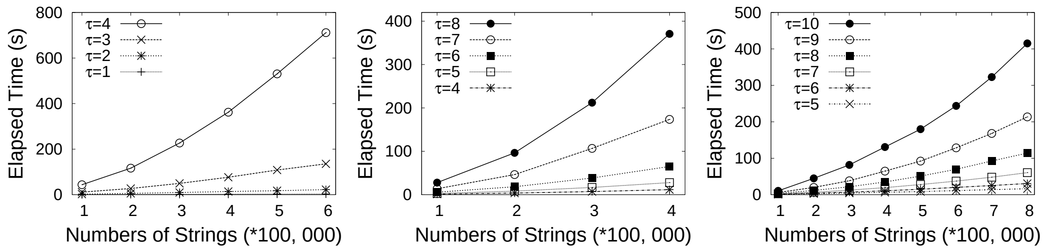
(a) Author (Avg Len = 15) (b) Query Log (Avg Len = 45) (c) Author+Title (Avg Len = 105)

Figure 14: Elapsed time for verification



(a) Author (Avg Len = 15) (b) Query Log (Avg Len = 45) (c) Author+Title (Avg Len = 105)

Figure 15: Comparison with state-of-the-art methods



(a) Author (Avg Len = 15) (b) Query Log (Avg Len = 45) (c) Author+Title (Avg Len = 105)

Figure 16: Scalability

them, especially for $\tau \geq 2$. The main reason is as follows. ED-JOIN must use a smaller q for a larger threshold. In this way ED-JOIN will involve large numbers of candidate pairs, since a smaller q has rather lower pruning power [23]. TRIE-JOIN used the prefix filtering to find similar pairs using a trie structure. If a small number of strings shared prefixes, TRIE-JOIN had low pruning power and was expensive to traverse the trie structure. Instead our framework utilized segments to prune large numbers of dissimilar pairs. The segments were selected across the strings and not restricted to prefix filtering. For instance, for τ=4, TRIE-JOIN took 2500 seconds. PASS-JOIN improved it to 700 seconds. ED-JOIN was rather slow and even larger than 10,000 seconds.

On the Author+Title dataset with long strings, our method significantly outperformed ED-JOIN and TRIE-JOIN, even in 2-3 orders of magnitude. This is because TRIE-JOIN was rather expensive to traverse the trie structures with long strings, especially for large thresholds. ED-JOIN needed to used a mismatch technique to do pruning which was inefficient while our filtering algorithm is very efficient. In addition, our verification method was more efficient than existing ones. For instance, for $\tau = 8$, TRIE-JOIN took 15,000 seconds, ED-JOIN decreased it to 5000 seconds, and PASS-JOIN improved the time to 130 seconds.

In addition, we compared index sizes on three datasets, as shown in Table 3. We can observe that existing methods involve larger indices than our method. For example, on the Author+Title dataset, ED-JOIN had 335 MB index, TRIE-JOIN used 90 MB, and our method only took 2.1 MB. There are two main reasons. Firstly for each string with length l, ED-JOIN generated $l - q + 1$ grams where q is the gram length, and our method only generated $\tau + 1$ segments. Secondly for a string with length l, we only maintained the indices for strings with lengths between $l - \tau$ and l, and ED-JOIN kept indices for all strings. TRIE-JOIN needed to use a trie structure to maintain strings, which had overhead to store the strings (e.g., pointers to children and indices for searching children with a given character).

Table 3: Index sizes (MB)

Data Sets	Data Sizes	ED-JOIN ($q = 4$)	TRIE-JOIN	PASS-JOIN ($\tau = 4$)
Author	8.7	25.34	16.32	1.92
Query Log	20	72.17	69.65	4.96
Author+Title	88	335.24	90.17	2.1

6.4 Scalability

In this section, we tested the scalability of our method. We varied the number of strings in the dataset and tested the elapsed time. Figure 16 shows the results. We can see that our method achieved nearly linear scalability. For example, for $\tau = 4$, on the Author dataset, the elapsed time for 400,000 strings, 500,000 strings, and 600, 000 strings were respectively 360 seconds, 530 seconds, and 700 seconds.

7. RELATED WORK

There have been many studies on string similarity joins [7, 2, 3, 6, 18, 23, 24, 19]. The approaches most related to ours are TRIE-JOIN [20], All-Pairs-Ed [3], ED-JOIN [23], and Part-Enum [2]. All-Pairs-Ed is a q-gram-based method. It first generates q-grams for each string and then selects the first $q\tau + 1$ grams as a gram prefix based on a pre-defined order. It prunes the string pairs with no common grams and verifies the survived string pairs. ED-JOIN improves All-Pairs-Ed using location-based and content-based mismatch filter by decreasing the number of grams. It has been shown that ED-JOIN outperforms All-Pairs-Ed [3]. TRIE-JOIN uses a trie structure to do similarity joins using prefix filtering. Part-Enum proposed an effective signature scheme called Part-Enum to do similar joins for hamming distance. It has been proved that All-Pairs-Ed and Part-Enum are worse than ED-JOIN and TRIE-JOIN [20]. Thus we only compared with ED-JOIN and TRIE-JOIN.

Gravano et al. [7] proposed gram-based methods and used SQL statements for similarity joins inside relational databases. Sarawagi et al. [18] proposed inverted index-based algorithms to solve similarity-join problem. Chaudhuri et al. [6] proposed a primitive operator for effective similarity joins. Arasu et al. [2] developed a signature scheme which can be used as a filter for effective similarity joins. Xiao et al. [25] proposed ppjoin to improve all-pair algorithm by introducing positional filtering and suffix filtering. Xiao et al. [24] studied top-k similarity joins, which can directly find the top-k string pairs without a given threshold.

In addition, Jacox et al. [11] studied the metric-space similarity join. As this method is not as efficient as ED-JOIN and TRIE-JOIN [20], we did not compare with it in the paper. Chaudhuri et al. [6] proposed the prefix-filtering signature scheme for effective similarity join. Recently, Wang et al. [21] devised a new similarity function by tolerating token errors in token-based similarity and developed effective algorithms to support similarity join on such functions.

The other related studies are approximate string searching [5, 14, 8, 9, 26], which given a query string and a set of strings, finds all similar strings of the query string in the string set. Navarro studied the approximate string matching problem [17], which given a query string and a text string, finds all substrings of the text string that are similar to the query string. These two problems are different from our similarity-join problem, which given two sets of strings, finds all similar string pairs. There are some studies on selectivity estimation of approximate string queries [10, 12, 13] and approximate entity extraction [1, 4, 22, 15].

8. CONCLUSION

In this paper, we have studied the problem of string similarity joins with edit-distance constraints. We propose a partition-based method to do efficient similarity joins. We first sort strings and then visit strings in order. We build inverted indices for the visited strings. For each string, we select some of its substrings and utilize the selected substrings to find similar string pairs using the inverted indices. We develop a position-aware method and a multi-match-aware method to select substrings. We prove that the multi-match-aware selection method can minimize the number of selected substrings. We also develop efficient techniques to verify candidate pair based on length difference. We propose an extension-based method and share the computations on common prefixes to further improve the verification performance. Experiments show that our method outperforms state-of-the-art studies on both short strings and long strings.

9. ACKNOWLEDGEMENT

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APPENDIX

A. PROOF OF LEMMA 1

LEMMA 1. *Given a string r with $\tau+1$ segments and a string s, if s is similar to r within threshold τ, s must contain a substring which matches a segment of r.*

PROOF. We prove it by contradiction. Suppose string s contains no substring which matches a segment of string r. In other words, any segment of r will not match any substring of s. Thus for any transformation $\mathcal{T}$ from r to s, in each segment of r there at least exists an edit operation. That is in any transformation $\mathcal{T}$ there are at least $\tau+1$ edit operations. This contradicts that s is similar to r. Thus s must contain a substring which matches a segment of r. $\square$

B. PROOF OF THEOREM 1

THEOREM 1. *The position-aware substring selection method satisfies completeness.*

PROOF. For any string s, consider a string r with length $l(|s|-\tau \le l \le |s|)$ which is similar to s and visited before s. Consider any transformation $\mathcal{T}$ from s to r with $|\mathcal{T}| \le \tau$ edit operations. Based on Lemma 1, s must have a substring s_m matching a segment r_m of r in the transformation $\mathcal{T}$. We split r (s) into three parts: the left part before the matching segment r_l (s_l), the matching segment r_m (s_m), and the right part after the matching segment r_r (s_r). Suppose r_m is the i-th segment of r. Thus $r \in \mathcal{L}_l^i(r_m)$. Next we prove that $s_m \in \mathcal{W}_p(s, \mathcal{L}_l^i) \subseteq \mathcal{W}_p(s, l)$.

Firstly as $s_m = r_m$, $|s_m| = |r_m| = l_i$. Suppose the start position of s_m in s is p. Next we only need to prove that $p \in [p_{min}, p_{max}]$. As $[p_{min}, p_{max}] = [1, |s|-l_i+1] \cap [p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor, p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor]$, we only need to prove that $p \in [1, |s|-l_i+1]$ and $p \in [p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor, p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor]$.

Case 1: $p \in [1, |s|-l_i+1]$. Obviously, based on the boundary, for any substring, the minimal start position is 1. As the length of s_m is l_i, the maximal start position is $|s|-l_i+1$. Thus p must be in $[1, |s|-l_i+1]$.

Case 2: $p \in [p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor, p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor]$. We prove it by contradiction. Suppose $p \notin [p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor, p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor]$. As $\mathcal{T}$ transforms s_l to r_l, matches s_m with r_m, and transforms s_r to r_r. We have $\tau \ge |\mathcal{T}| \ge \text{ED}(s_l, r_l) + \text{ED}(s_m, r_m) + \text{ED}(s_r, r_r) \ge |p_i - p| + 0 + \Big|(|r|-p_i) - (|s|-p)\Big|$.

If $p < p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor$, we have

$$\begin{aligned}|\mathcal{T}| &\ge |p_i - p| + \big|(|r|-p_i) - (|s|-p)\big| \\ &\ge (\lfloor\frac{\tau-\triangle}{2}\rfloor + 1) + (\triangle + \lfloor\frac{\tau-\triangle}{2}\rfloor + 1) \\ &\ge (2\lfloor\frac{\tau-\triangle}{2}\rfloor + 1) + (\triangle + 1) \\ &\ge \tau - \triangle + (\triangle + 1) \ge \tau + 1 > \tau\end{aligned}$$

If $p > p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor$, we have

$$\begin{aligned}|\mathcal{T}| &\ge |p_i - p| + \big|(|r|-p_i) - (|s|-p)\big| \\ &\ge (\lfloor\frac{\tau+\triangle}{2}\rfloor + 1) + (\lfloor\frac{\tau+\triangle}{2}\rfloor + 1 - \triangle) \\ &\ge (2\lfloor\frac{\tau+\triangle}{2}\rfloor + 1) + (1 - \triangle) \\ &\ge \tau + \triangle + (1 - \triangle) \ge \tau + 1 > \tau\end{aligned}$$

In both cases, we have $|\mathcal{T}| > \tau$ which contradicts with $|\mathcal{T}| \le \tau$. Thus $p \in [p_i - \lfloor\frac{\tau-\triangle}{2}\rfloor, p_i + \lfloor\frac{\tau+\triangle}{2}\rfloor]$.

Based on Case 1 and Case 2, $p \in [p_{min}, p_{max}]$. Thus for any string r with length $l(|s|-\tau \le l \le |s|)$ which is similar to s and visited before s, r must have an i-th segment r_m that matches a substring $s_m \in \mathcal{W}_p(s, \mathcal{L}_l^i)$. $\square$

C. PROOF OF THEOREM 2

We first give a lemma to prove that the multi-match-aware selection method from the left-side perspective satisfies completeness.

LEMMA 2.1. *The multi-match-aware selection method from the left-side perspective satisfies completeness.*

PROOF. For any string s, consider a string r with length $l(|s|-\tau \le l \le |s|)$ which is similar to s and visited before s. Consider any transformation $\mathcal{T}$ from r to s with $|\mathcal{T}| \le \tau$ edit operations. Based on Lemma 1, s must have a substring s_m matching a segment r_m of r in the transformation $\mathcal{T}$. We assume that r_m is the *last* segment of r which matches a substring s_m of s in transformation $\mathcal{T}$. Without loss of generality, suppose the start position of s_m in s is p and r_m is the i-th segment of r. Thus $r \in \mathcal{L}_l^i(r_m)$. Based on Theorem 1, we have $|s_m| = l_i$ and $p \in [1, |s|-l_i+1]$.

As $[\bot_i^l, \top_i^l] = [1, |s|-l_i+1] \cap [p_i - (i-1), p_i + (i-1)]$, we only need to prove that $p \in [p_i - (i-1), p_i + (i-1)]$.

We prove it by contradiction. Suppose $p \notin [p_i - (i-1), p_i + (i-1)]$, we have $\text{ED}(s_l, r_l) \ge |p - p_i| \ge i$. As $\mathcal{T}$ transforms s_l to r_l, matches s_m with r_m, and transforms s_r to r_r, we have $\tau \ge |\mathcal{T}| \ge \text{ED}(s_l, r_l) + \text{ED}(s_r, r_r) \ge i + \text{ED}(s_r, r_r)$, thus $\text{ED}(s_r, r_r) \le \tau - i$. On the other hand, as there are $\tau + 1 - i$ segments in r_r, there must exist a segment in r_r which matches a substring of s_r based on Lemma 1. This contradicts with the assumption that r_m is the *last* segment of r which matches a substring of s. Thus $p \in [p_i - (i-1), p_i + (i-1)]$.

Therefore for any string r with length $l(|s|-\tau \le l \le |s|)$ which is similar to s and visited before s, r must have an i-th segment r_m matching a substring $s_m \in \mathcal{W}_r(s, \mathcal{L}_l^i)$. $\square$

Similarly, we can prove that the multi-match-aware selection method from the right-side perspective also satisfies completeness based on Lemma 2.1. Next we prove that the multi-match-aware selection method satisfies completeness. For each inverted index $\mathcal{L}_l^i$, this method selects a set $\mathcal{W}_m(s, \mathcal{L}_l^i)$ which is composed of the substrings of s with start positions in $[\bot_i, \top_i]$ and with length l_i, where $\bot_i = \max(\bot_i^l, \bot_i^r)$ and $\top_i = \min(\top_i^l, \top_i^r)$, $\bot_i^l = \max\big(1, p_i - (i-1)\big)$ and $\top_i^l = \min\big(|s|-l_i+1, p_i + (i-1)\big)$, and $\bot_i^r = \max\big(1, p_i + \triangle - (\tau+1-i)\big)$ and $\top_i^r = \min\big(|s|-l_i+1, p_i + \triangle + (\tau+1-i)\big)$. Then the method unions the sets to generate $\mathcal{W}_m(s, l)$. That is $\mathcal{W}_m(s, l) = \cup_{i=1}^{\tau+1} \mathcal{W}_m(s, \mathcal{L}_l^i)$.

THEOREM 2. *The multi-match-aware substring selection method satisfies completeness.*

PROOF. For any string s, consider a string r with length $l(|s|-\tau \le l \le |s|)$ which is similar to s and visited before s. Consider any transformation $\mathcal{T}$ from r to s with $|\mathcal{T}| \le \tau$ edit operations. For any substring s_m of s matching a segment r_m of r in $\mathcal{T}$, based on Theorem 1, s_m's start position p

must be in $[1, |s| - l_i + 1]$, and $|s_m| = l_i$ (suppose r_m is the i-th segment of r).

Next, we only need to prove that in the transformation $\mathcal{T}$, there exists an i-th segment r_m for r matching a substring of s such that $p \in [\max\big(p_i - (i-1), p_i + \triangle - (\tau + 1 - i)\big), \min\big(p_i + (i-1), p_i + \triangle + (\tau + 1 - i)\big)]$. That is we only need to prove that $p \in [p_i - (i-1), p_i + (i-1)]$ and $p \in [p_i + \triangle - (\tau + 1 - i), p_i + \triangle + (\tau + 1 - i)]$.

Based on Lemma 1, there must exist at least one substring of s that matches a segment of r in transformation $\mathcal{T}$. Consider the *first* segment r_m of r that matches a substring s_m of s in transformation $\mathcal{T}$. Without loss of generality, suppose the start position of s_m in s is p and r_m is the k-th segment of r. Thus $r \in \mathcal{L}_l^k(r_m)$. We split s (r) into three parts: the part before the matching segment s_l (r_l), the matching segment s_m (r_m), and the part after the matching segment s_r (r_r). Based on Lemma 2.1 (from the right-side perspective), $p \in [p_k + \triangle - (\tau+1-k), p_k + \triangle + (\tau+1-k)]$. If $p \in [p_k - (k-1), p_k + (k-1)]$, we set $i = k$ and the theorem is proved; otherwise suppose $p \notin [p_k - (k-1), p_k + (k-1)]$, we have $\text{ED}(s_l, r_l) \geq |p - p_k| \geq k$. As $\mathcal{T}$ transforms s_l to r_l, matches s_m with r_m, and transforms s_r to r_r, $\tau \geq |\mathcal{T}| \geq \text{ED}(s_l, r_l) + \text{ED}(s_r, r_r) \geq k + \text{ED}(s_r, r_r)$, thus $\text{ED}(s_r, r_r) \leq \tau - k$. On the other hand, as there are $\tau + 1 - k$ segments in r_r, there must exist a segment in r_r which matches a substring of s_r in transformation $\mathcal{T}$ based on Lemma 1.

Suppose r'_m is the *first* segment in r_r that matches a substring s'_m of s_r in transformation $\mathcal{T}$. Without loss of generality, suppose the start position of s'_m in s is p' and r'_m is the j-th($j > k$) segment of r. Thus $r \in \mathcal{L}_l^j(r'_m)$. We split s_r (r_r) into three parts: the part before the matching segment s'_l (r'_l), the matching segment s'_m (r'_m), and the part after the matching segment s'_r (r'_r). Next we prove that $p' \in [p_j + \triangle - (\tau+1-j), p_j + \triangle + (\tau+1-j)]$. We prove it by contradiction. Suppose $p' \notin [p_j + \triangle - (\tau+1-j), p_j + \triangle + (\tau+1-j)]$. We have

$$\begin{aligned}\text{ED}(r'_r, s'_r) &\geq |(|s| - p') - (|r| - p_j)| \\ &= |p_j + (|s| - l) - p'| \\ &= |(p_j + \triangle) - p'| \\ &\geq \tau + 1 - j + 1\end{aligned}$$

As $\mathcal{T}$ transforms s_l to r_l, matches s_m with r_m, transforms s'_l to r'_l, matches s'_m with r'_m, and transforms s'_r to r'_r, $\tau \geq |\mathcal{T}| \geq \text{ED}(r_l, s_l) + \text{ED}(r'_l, s'_l) + \text{ED}(r'_r, s'_r) \geq k + \text{ED}(r'_l, s'_l) + \tau + 1 - j + 1$, thus $\text{ED}(r'_l, s'_l) \leq \tau - k - (\tau + 1 - j + 1) = j - k - 2$. On the other hand, as there are $j-k-1$ segments in r'_l, there must exist a segment of r'_l which matches a substring of s'_l in the transformation $\mathcal{T}$ based on Lemma 1. This contradicts with the assumption that r'_m is the *first* segment in r_r that matches a substring of s_r. Thus $p' \in [p_j + \triangle - (\tau + 1 - j), p_j + \triangle + (\tau + 1 - j)]$. If $p' \in [p_j - (j-1), p_j + (j-1)]$, we set $i = j$ and the theorem is proved; otherwise, we have $p' \in [p_j + \triangle - (\tau + 1 - j), p_j + \triangle + (\tau + 1 - j)]$ and $p' \notin [p_j - (j-1), p_j + (j-1)]$. We can repeat our above proof until the theorem is proved or reaching the *last* segment of r (denoted by r''_m) that matches a substring of s (denoted by s''_m). In the latter case, we have r''_m is the i-th segment of r and the start position of s''_m is p''. Based on the above proof, we have $p'' \in [p_i + \triangle - (\tau+1-i), p_i + \triangle + (\tau+1-i)]$. Based on the proof in Lemma 2.1, we have $p'' \in [p_i - (i-1), p_i + (i-1)]$. Thus $p'' \in [p_i - (i-1), p_i + (i-1)] \cap [p_i + \triangle - (\tau + 1 - i), p_i + \triangle + (\tau + 1 - i)] = [\perp_i, \top_i]$.

In summary, for any string r with length $l(|s| - \tau \leq l \leq |s|)$ which is similar to s and visited before s, r must have an i-th segment r_m matching a substring $s_m \in \mathcal{W}_m(s, \mathcal{L}_l^i)$. $\square$

D. PROOF OF LEMMA 2

LEMMA 2. $|\mathcal{W}_m(s, l)| = \lfloor \frac{\tau^2 - \triangle^2}{2} \rfloor + \tau + 1$.

PROOF. As $\mathcal{W}_m(s, l) = \cup_{i=1}^{\tau+1} \mathcal{W}_m(s, \mathcal{L}_l^i)$, we have

$$|\mathcal{W}_m(s, l)| = \sum_{i=1}^{\tau+1} |\mathcal{W}_m(s, \mathcal{L}_l^i)| = \sum_{i=1}^{\tau+1} (\top_i - \perp_i + 1) =$$

$$\sum_{i=1}^{\tau+1} \Big(\min\big(|s| - l_i + 1, p_i + (i-1), p_i + \triangle + (\tau+1-i)\big) -$$

$$\max\big(1, p_i - (i-1), p_i + \triangle - (\tau + 1 - i)\big) + 1\Big)$$

As $(p_{i+1} - (i+1)) - (p_i - i) = p_{i+1} - p_i - 1 \geq 0$, $p_i - i$ is a monotonically increasing function. Thus for any $i \in [1, \tau + 1]$, we have $p_i - (i-1) \geq p_1 - (1-1) = 1$ and $p_i + \triangle + (\tau+1-i) = p_i - i + \triangle + \tau + 1 \leq p_{\tau+1} - (\tau+1) + \triangle + \tau + 1 = p_{\tau+1} + \triangle = p_{\tau+1} + |s| - |r| = p_{\tau+1} + |s| - (p_{\tau+1} + l_{\tau+1}) = |s| - l_{\tau+1} \leq |s| - l_i < |s| - l_i + 1$, thus

$$|\mathcal{W}_m(s, l)| = \sum_{i=1}^{\tau+1} (\top_i - \perp_i + 1) =$$

$$\sum_{i=1}^{\tau+1} \Big(\min\big(p_i + (i-1), p_i + \triangle + (\tau + 1 - i)\big) -$$

$$\max\big(p_i - (i-1), p_i + \triangle - (\tau + 1 - i)\big) + 1\Big)$$

Consider $\perp_i = \max\big(p_i - (i-1), p_i + \triangle - (\tau + 1 - i)\big)$. If $p_i - (i-1) \geq p_i + \triangle - (\tau + 1 - i)$, we have $\perp_i = p_i - (i-1)$. In this case $i \leq \lfloor \frac{\tau - \triangle}{2} \rfloor + 1$. On the contrary, if $p_i - (i-1) < p_i + \triangle - (\tau + 1 - i)$, we have $\perp_i = p_i + \triangle - (\tau + 1 - i)$ for $i > \lfloor \frac{\tau - \triangle}{2} \rfloor + 1$.

Similarly, for $\top_i = \min\big(p_i + (i-1), p_i + \triangle + (\tau + 1 - i)\big)$, if $p_i + (i-1) \leq p_i + \triangle + (\tau + 1 - i)$, we have $\top_i = p_i + (i-1)$. In this case $i \leq \lfloor \frac{\tau + \triangle}{2} \rfloor + 1$. On the contrary if $p_i + (i-1) > p_i + \triangle + (\tau + 1 - i)$, we have $\top_i = p_i + \triangle + (\tau + 1 - i)$ for $i > \lfloor \frac{\tau + \triangle}{2} \rfloor + 1$.

In this way, to compute $\perp_i$-$\top_i$+1, we split $i \in [1, \tau+1]$ into $i \leq \lfloor \frac{\tau - \triangle}{2} \rfloor + 1$, $\lfloor \frac{\tau - \triangle}{2} \rfloor + 2 \leq i \leq \lfloor \frac{\tau + \triangle}{2} \rfloor + 1$, and $\lfloor \frac{\tau + \triangle}{2} \rfloor + 2 \leq i \leq \tau + 1$.

$$|\mathcal{W}_m(s, l)| = \sum_{i=1}^{\tau+1} (\top_i - \perp_i + 1) =$$

$$\sum_{i=1}^{\lfloor \frac{\tau - \triangle}{2} \rfloor + 1} \Big(\big(p_i + (i-1) - (p_i - (i-1)\big) + 1\Big) +$$

$$\sum_{i=\lfloor \frac{\tau - \triangle}{2} \rfloor + 2}^{\lfloor \frac{\tau + \triangle}{2} \rfloor + 1} \Big(\big(p_i + (i-1)) - (p_i + \triangle - (\tau + 1 - i)\big) + 1\Big) +$$

$$\sum_{i=\lfloor \frac{\tau + \triangle}{2} \rfloor + 2}^{\tau+1} \Big(\big(p_i + \triangle + (\tau+1-i)) - (p_i + \triangle - (\tau+1-i)\big) + 1\Big)$$

$$= \sum_{i=1}^{\lfloor \frac{\tau - \triangle}{2} \rfloor + 1} (2i - 1) + \sum_{i=\lfloor \frac{\tau - \triangle}{2} \rfloor + 2}^{\lfloor \frac{\tau + \triangle}{2} \rfloor + 1} (\tau - \triangle + 1) + \sum_{i=\lfloor \frac{\tau + \triangle}{2} \rfloor + 2}^{\tau+1} (2\tau - 2i + 3)$$

$$= \tau^2 + \triangle\tau - \triangle^2 + \triangle + 1 + \lfloor \frac{\tau + \triangle}{2} \rfloor^2 + \lfloor \frac{\tau - \triangle}{2} \rfloor^2 + 2\lfloor \frac{\tau - \triangle}{2} \rfloor - 2\tau \lfloor \frac{\tau + \triangle}{2} \rfloor$$

If $\tau+\triangle$ is even, $\tau-\triangle$ must be even. Thus $\lfloor\frac{\tau-\triangle}{2}\rfloor=\frac{\tau-\triangle}{2}$ and $\lfloor\frac{\tau+\triangle}{2}\rfloor=\frac{\tau+\triangle}{2}$, and we have

$$|\mathcal{W}_m(s,l)|=\frac{\tau^2-\triangle^2}{2}+\tau+1=\lfloor\frac{\tau^2-\triangle^2}{2}\rfloor+\tau+1$$

If $\tau+\triangle$ is odd, $\tau-\triangle$ must be odd. Thus $\lfloor\frac{\tau-\triangle}{2}\rfloor=\frac{\tau-\triangle-1}{2}$ and $\lfloor\frac{\tau+\triangle}{2}\rfloor=\frac{\tau+\triangle-1}{2}$, and we have

$$|\mathcal{W}_m(s,l)|=\frac{\tau^2-\triangle^2+1}{2}+\tau=\lfloor\frac{\tau^2-\triangle^2}{2}\rfloor+\tau+1$$

Thus the lemma is proved. $\square$

E. PROOF OF LEMMA 3

LEMMA 3. *Let $\mathcal{W}_\ell(s,l),\mathcal{W}_f(s,l),\mathcal{W}_p(s,l),\mathcal{W}_m(s,l)$ respectively denote the set of selected substrings using the length-based selection method, the shift-based selection method, the position-aware selection method, and the multi-match-aware selection method. For any string s and a length l, we have*

$$\mathcal{W}_m(s,l)\subseteq\mathcal{W}_p(s,l)\subseteq\mathcal{W}_f(s,l)\subseteq\mathcal{W}_\ell(s,l).$$

PROOF. If $\tau=0$, the four methods select s as its selected substring. Thus $\mathcal{W}_\ell(s,l)=\mathcal{W}_f(s,l)=\mathcal{W}_p(s,l)=\mathcal{W}_m(s,l)=\{s\}$. Next we prove the lemma for $\tau>0$.

Given $\mathcal{L}_l^i$, firstly the substring length of each method is the same, i.e., l_i. Next we consider the start positions.

(i) **We first prove $\mathcal{W}_f(s,l)\subseteq\mathcal{W}_\ell(s,l)$.**

For $\mathcal{W}_\ell(s,l)$, the start positions are in $[1,|s|-l_i+1]$.

For $\mathcal{W}_f(s,l)$, the start positions are in

$$[\max(1,p_i-\tau),\min(|s|-l_i+1,p_i+\tau)].$$

To prove $\mathcal{W}_f(s,l)\subseteq\mathcal{W}_\ell(s,l)$, we only need to prove

$$[1,|s|-l_i+1]\supseteq[\max(1,p_i-\tau),\min(|s|-l_i+1,p_i+\tau)].$$

It is obvious that $\max(1,p_i-\tau)\geq 1$ and $\min(|s|-l_i+1,p_i+\tau)\leq|s|-l_i+1$. Thus we have $\mathcal{W}_f(s,l)\subseteq\mathcal{W}_\ell(s,l)$.

(ii) **We then prove $\mathcal{W}_p(s,l)\subseteq\mathcal{W}_f(s,l)$.**

For $\mathcal{W}_p(s,l)$, the start positions are in

$$\big[\max(1,p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor),\min(|s|-l_i+1,p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor)\big]=$$
$$\big[p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor,p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor\big]\cap[1,|s|-l_i+1]$$

For $\mathcal{W}_f(s,l)$, as $[\max(1,p_i-\tau),\min(|s|-l_i+1,p_i+\tau)]=[p_i-\tau,p_i+\tau]\cap[1,|s|-l_i+1]$, to prove $\mathcal{W}_p(s,l)\subseteq\mathcal{W}_f(s,l)$, we only need to prove

$$[p_i-\tau,p_i+\tau]\supseteq[p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor,p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor]$$

As $0\leq\triangle\leq\tau$, we have $p_i-\tau\leq p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor$ and $p_i+\tau\geq p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor$. Thus $\mathcal{W}_p(s,l)\subseteq\mathcal{W}_f(s,l)$.

(iii) **Next we prove $\mathcal{W}_m(s,l)\subseteq\mathcal{W}_p(s,l)$.**

For $\mathcal{W}_m(s,l)$, the start positions are in

$$\begin{aligned}&[\bot_i,\top_i]=[\max(\bot_i^l,\bot_i^r),\min(\top_i^l.\top_i^r)]=\\&\quad[\max(1,p_i-(i-1),p_i+\triangle-(\tau+1-i)),\\&\quad\min(|s|-l_i+1,p_i+(i-1),p_i+\triangle+(\tau+1-i))]=\\&\quad[\max(p_i-(i-1),p_i+\triangle-(\tau+1-i)),\\&\quad\min(p_i+(i-1),p_i+\triangle+(\tau+1-i)]\cap[1,|s|-l_i+1]\end{aligned}$$

To prove $\mathcal{W}_m(s,l)\subseteq\mathcal{W}_p(s,l)$, we only need to prove

$$[p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor,p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor]\supseteq[\bot_i,\top_i].$$

Firstly we prove $\bot_i=\max\big(p_i-(i-1),p_i+\triangle-(\tau+1-i)\big)\geq p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor$. If $p_i-(i-1)\geq p_i+\triangle-(\tau+1-i)$, we have $\bot_i=p_i-(i-1)$. In this case $i\leq\lfloor\frac{\tau-\triangle}{2}\rfloor+1$. That is $i-1\leq\lfloor\frac{\tau-\triangle}{2}\rfloor$. Obviously $\bot_i=p_i-(i-1)\geq p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor$. On the contrary, if $p_i-(i-1)<p_i+\triangle-(\tau+1-i)$, we have $\bot_i=p_i+\triangle-(\tau+1-i)$. In this case $i>\lfloor\frac{\tau-\triangle}{2}\rfloor+1$. That is $i-1>\lfloor\frac{\tau-\triangle}{2}\rfloor$. As $\bot_i=p_i+\triangle-(\tau+1-i)=p_i+(i-1)-(\tau-\triangle)$, $\bot_i\geq p_i-\lfloor\frac{\tau-\triangle}{2}\rfloor$.

Then we prove that $\top_i=\min\big(p_i+(i-1),p_i+\triangle+(\tau+1-i)\big)\leq p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor$. If $p_i+(i-1)\leq p_i+\triangle+(\tau+1-i)$, we have $\top_i=p_i+(i-1)$. In this case $i\leq\lfloor\frac{\tau+\triangle}{2}\rfloor+1$. That is $i-1\leq\lfloor\frac{\tau+\triangle}{2}\rfloor$. Obviously $\top_i=p_i+(i-1)\leq p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor$. On the contrary if $p_i+(i-1)>p_i+\triangle+(\tau+1-i)$, we have $\top_i=p_i+\triangle+(\tau+1-i)$. In this case $i>\lfloor\frac{\tau+\triangle}{2}\rfloor+1$. That is $i-1>\lfloor\frac{\tau+\triangle}{2}\rfloor$. As $\top_i=p_i+\triangle+(\tau+1-i)=p_i-(i-1)+\tau+\triangle$, $\top_i\leq p_i+\lfloor\frac{\tau+\triangle}{2}\rfloor$.

Thus we have $\mathcal{W}_m(s,l)\subseteq\mathcal{W}_p(s,l)$.

Therefore $\mathcal{W}_m(s,l)\subseteq\mathcal{W}_p(s,l)\subseteq\mathcal{W}_f(s,l)\subseteq\mathcal{W}_\ell(s,l)$ and the lemma is proved. $\square$

F. PROOF OF THEOREM 3

We first prove that the substring set $\mathcal{W}_m(s,l)$ generated by the multi-match-aware selection method has the minimum size. That is for any other substring set $\mathcal{W}(s,l)$ generated by a method that satisfies completeness, we have $|\mathcal{W}_m(s,l)|\leq|\mathcal{W}(s,l)|$. The basic idea of our proof is as follows. For each substring $s_m\in\mathcal{W}_m(s,l)$, we generate a substring set $\Phi(s_m,l)$, such that

Condition (1): If a substring selection method satisfies completeness, it must select a substring in $\Phi(s_m,l)$;

Condition (2): For any two substrings $s_m\neq s_{m'}$ in $\mathcal{W}_m(s,l)$, if a substring selection method satisfies completeness, it must select a substring in $\Phi(s_m,l)$ and another substring in $\Phi(s_{m'},l)$, and the two substrings are not the same. Notice that two selected substrings are said to be the same, if they are selected for the same segment and have the same start positions and lengths.

Obviously if we can generate a substring set $\Phi(s_m,l)$ satisfying the above two conditions, we have $|\mathcal{W}_m(s,l)|\leq|\mathcal{W}(s,l)|$ (See the proof in Theorem 3). Next we discuss how to generate the substring set $\Phi(s_m,l)$.

Notice that based on the definition of completeness, we need to guarantee completeness for every string with length $|s|$, thus $\mathcal{W}_m(s,l)$ does not depend on the content of s. In other words, the size of $\mathcal{W}_m(s,l)$ only depends on $|s|$. Without loss of generality, we consider a string s whose characters are distinct, i.e., $s[i] \neq s[j]$ for $i \neq j$ where $s[i]$ is the i-th character of s for $1 \leq i \leq |s|$.

Next we construct a string r with length l based on s, such that (1) r is similar to s with τ edit operations; (2) if a substring selection method that satisfies completeness does not select a substring from $\Phi(s_m,l)$, the method will miss the similar pair $\langle s,r\rangle$.

We generate r from s as follows. Suppose the length of r is l, and the start position of the i-th segment is p_i and the length is l_i. We first partition s into $\tau+1$ substrings and then use the k-th substring of s to generate the k-th segment of r. Let l'_k and p'_k respectively denote the length and the start position of the k-th substring of s. We use p_k and l_k to deduce l'_k and p'_k. Obviously we have $p'_1 = 1$ and $p'_k = p'_1 + \sum_{j=1}^{k-1} l'_j$. Next we focus on how to get the length of each substring of s (l'_k) and how to generate a segment of r as follows.

Suppose s_m is selected from the i-th segment, i.e., $s_m \in \mathcal{W}_m(s,\mathcal{L}^i_l)$, and the position of s_m in s is p. Based on the multi-match-aware selection method, as $p \in [p_i-(i-1), p_i+(i-1)] \cap [p_i+\triangle-(\tau+1-i), p_i+\triangle+(\tau+1-i)]$, we can easily deduce that $|p-p_i| \leq i-1$ and $|(l-p_i)-(|s|-p)| = |p-p_i-\triangle| \leq \tau+1-i$. We first consider $p \leq p_i$.

For each $k \in [1, p_i-p]$, we generate the k-th segment of r from the k-th substring of s by applying an insertion operation. As an insertion operation will increase the length by 1, we have $l'_k = l_k - 1$. Notice that as $p_i - p \leq i-1$ (based on the multi-match-aware selection), we can choose p_i-p substrings from the first $i-1$ substrings of s (before s_m) to apply an insertion operation on each substring. Here we choose the first p_i-p substrings.

For each $k \in [p_i-p+1, i-1]$, we generate the k-th segment of r from the k-th substring of s by applying a substitution operation. As a substitution operation will not change the substring length, we have $l'_k = l_k$.

For $k = i$, the i-th segment of r is exactly the i-th substring of s (i.e., s_m), we have $l'_k = l_k$.

For each $k \in [i+1, \tau+1-(p_i-p+\triangle)]$, we generate the k-th segment of r from the k-th substring of s by applying a substitution operation. As a substitution operation will not change the substring length, we have $l'_k = l_k$.

For each $k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]$, we generate the k-th segment of r from the k-th substring of s by applying a deletion operation. As a deletion operation will decrease the length by 1, we have $l'_k = l_k + 1$. Notice that to make the length of r be l, we need to do some deletions on the last $\tau+1-i$ substrings of s (after s_m). As $l = |s| - \triangle$, we need to do $p_i-p+\triangle$ deletions. As $p_i-p+\triangle \leq \tau+1-i$ (based on the multi-match-aware selection), we can choose $p_i-p+\triangle$ substrings from the the last $\tau+1-i$ substrings of s. Here we choose the last $p_i-p+\triangle$ substrings of s to apply a deletion operation for each substring.

Obviously, we only do τ edit operations on s to generate r. Thus r is similar to s with τ edit operations. Next we discuss how to apply the insertion, substitution, and deletion operations.

For $k \in [1, p_i-p]$, we do insertion operations. As we can use a special character that does not appear in s to apply the insertion operation, we do not need to select any substring of s for the k-th segment of r. This is because the k-th segment of r will not match any substring of s as it contains a special character.

For $k \in [p_i-p+1, i-1]$, we need to do a substitution operation on each substring of s. Similarly, we can use a special character that does not appear in s to apply the substitution operation, thus we also do not select any substring for the k-th segment of r.

For $k = i$, as s_m matches r_m, we add s_m into $\Phi(s_m,l)$.

For $k \in [i+1, \tau+1-(p_i-p+\triangle)]$, we need to do a substitution operation on each substring of s. Similarly, we can use a special character that does not appear in s to apply the substitution operation, thus we also do not select any substring for the k-th segment of r.

Finally for each $k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]$, we need to do deletion operations. If each substring has no smaller than 3 characters, we generate the k-th segment of r from the the k-th substring of s by deleting a middle (i.e. neither the first nor the last) character of the substring to apply the deletion operation. In this case we do not need to select such substrings for the k-th segment of r. This is because as the characters in the substrings are distinct, if we delete a middle character of the substring, the generated segment will not match any substring of s. Thus if each substring has no smaller that 3 characters, $\Phi(s_m,l) = \{s_m\}$. Similarly, if $p \geq p_i+\triangle$ or $p_i+\triangle > p > p_i$, $\Phi(s_m,l) = \{s_m\}$ (each substring has no smaller than 3 characters). In this case, we can easily prove that $\Phi(s_m,l)$ satisfies condition (1) as formalized in Lemma 3.1.

Next we consider the case that some substrings have less than 3 characters.

If $p \leq p_i$, for the last $p_i-p+\triangle$ substrings of s, instead of deleting a character in each substring, we delete the last $p_i-p+\triangle$ characters of s. In this way, we set the k-th segment $r[p_k,l_k]$ of string r as $s[|s|-(p_i-p+\triangle)-(|r|-p_k), l'_k = l_k]$. We give the basic idea as follows. Consider the k-th segment $r[p_k,l_k]$ of r for $k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]$. Let $s[x_k \cdots |s|]$ denote the substring of s from the x_k-th character to the end of s and $r[p_k \cdots |r|]$ denote the substring of r from the p_k-th character to the end of r. As we delete the last $p_i-p+\triangle$ characters of s to make $s[x_k \cdots |s|]$ and $r[p_k \cdots |r|]$ have the same length, we have $|s[x_k \cdots |s|]| - |r[p_k \cdots |r|]| = p_i-p+\triangle$. That is $x_k = |s|-(p_i-p+\triangle)-(|r|-p_k)$. Thus $r[p_k,l_k] = s[x_k, l'_k = l_k]$ for $k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]$. Note that the k-th segment of r will not match any other substring of s as the characters of s are distinct. Thus $\Phi(s_m,l) = \{s_m\} \cup \{s[|s|-(p_i-p+\triangle)-(|r|-p_k), l_k] | k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]\}$. We can easily prove that $\Phi(s_m,l)$ satisfies condition (1) as formalized in Lemma 3.1.

Similarly, if $p \geq p_i+\triangle$, we do deletion operations on the first $p-p_i$ substrings of s. We delete the first $p-p_i$ characters of s, and we have $x_k = p_k + (p-p_i)$ and $l'_k = l_k$ for $k \in [1, p-p_i]$. Thus $\Phi(s_m,l) = \{s_m\} \cup \{s[p_k+(p-p_i), l_k] | k \in [1, p-p_i]\}$. We can easily prove that $\Phi(s_m,l)$ satisfies condition (1) as formalized in Lemma 3.1.

If $p_i < p < p_i+\triangle$, we do deletion operations on the first $p-p_i$ substrings and the last $p_i-p+\triangle$ substrings. We delete the first $p-p_i$ characters of s and the last $p_i-p+\triangle$ characters of s, and we have $x_k = p_k+(p-p_i)$ and $l'_k = l_k$ for $k \in [1, p-p_i]$, and $x_k = |s|-(p_i-p+\triangle)-(|r|-p_k)$ and $l'_k = l_k$ for $k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]$. In this case, $\Phi(s_m,l) = \{s_m\} \cup \{s[p_k+(p-p_i), l_k] | k \in [1, p-p_i]\} \cup \{s[|s|-(p_i-p+\triangle)-(|r|-p_k), l_k] | k \in [\tau+1-(p_i-p+\triangle)+1, \tau+1]\}$. We can easily prove that $\Phi(s_m,l)$ satisfies condition (1) as formalized in Lemma 3.1.

LEMMA 3.1. *If a selection method satisfies completeness, it must select a substring in $\Phi(s_m, l)$ for each $s_m \in \mathcal{W}_m(s, l)$.*

PROOF. Firstly if each substring of s has at least 3 characters, $\Phi(s_m, l) = \{s_m\}$. As only s_m matches a segment of string r, if a substring selection method does not select s_m, the method must miss a similar pair $\langle s, r\rangle$. Thus any selection method that satisfies completeness must select a substring in $\Phi(s_m, l)$.

Secondly, if some substrings of s have less than 3 characters and we need to do deletion operations on such substrings. We consider the following three cases.

Case (1): $p \le p_i$. $\Phi(s_m, l) = \{s_m\} \cup \{s[|s| - (p_i - p + \triangle) - (|r| - p_k), l_k]\}$. As only substrings in $\Phi(s_m, l)$ matches a segment of string r. If a substring selection method does not select a substring, the method must miss a similar pair $\langle s, r\rangle$. Thus any selection method that satisfies completeness must select a substring in $\Phi(s_m, l)$.

Case (2): $p \ge p_i + \triangle$. It is similar to Case (1).

Case (3): $p_i<p<p_i+\triangle$. It is similar to Case (1) and Case (2).

Thus the lemma is proved. □

Then we prove that for any two substrings $s_m \ne s_{m'}$ in $\mathcal{W}_m(s, l)$, if a substring selection method satisfies completeness, it must contain a substring in $\Phi(s_m, l)$ and another substring in $\Phi(s_{m'}, l)$ such that the two substrings are not the same as formalized in Lemma 3.2.

LEMMA 3.2. *For any two substrings $s_m \ne s_{m'}$ in $\mathcal{W}_m(s, l)$, if a method satisfies completeness, it must contain a substring in $\Phi(s_m, l)$ and another substring in $\Phi(s_{m'}, l)$, and the two substrings are not the same.*

PROOF. Without loss of generality, suppose $s_m \in \mathcal{W}_m(s, \mathcal{L}_l^i) \subseteq \mathcal{W}_m(s, l)$ with start position p and $s'_m \in \mathcal{W}_m(s, \mathcal{L}_l^j) \subseteq \mathcal{W}_m(s, l)$ with start position p' (Notice that i may be equal to j). We only need to prove that, for any $s_k \in \Phi(s_m, l)$ and $s'_k \in \Phi(s_{m'}, l)$, (1) $s_k \ne s'_k$; or (2) If $s_k = s'_k$, a method that only selects s_k (or s'_k) from $\Phi(s_m, l)$ and $\Phi(s_{m'}, l)$ will miss a similar pair.

Firstly, if s_k and s'_k are selected for different segments, we have $s_k \ne s'_k$ and the lemma is proved. Secondly, s_k and s'_k are selected for the same segment. Without loss of generality, suppose they are selected for the k-th segment of r. We consider the following cases.

Case 1: $s_k = s_m$ and $s'_k = s'_m$. In this case as $s_m \ne s'_m$, $s_k \ne s'_k$.

Case 2: $s_k = s[|s|-(p_i-p+\triangle)-(|r|-p_k), l_k]$ and $s'_k = s'_m$. In this case $p < p_i + \triangle$. We prove that $s_k \ne s'_k$ as follows. If $k \ne j$, $s_k \ne s'_k$, as s_k is selected for the k-th segment and s'_k is selected for j-th segment. If $k = j$, the start positions of $s'_k \in \mathcal{W}_m(s, \mathcal{L}_l^k)$ are in $[p_k + \triangle - (\tau + 1 - k), p_k + \triangle + (\tau + 1 - k)] \cap [p_k - (k - 1), p_k + (k - 1)]$. We can deduce $|s| - (p_i - p + \triangle) - (|r| - p_k) < p_k + \triangle - (\tau + 1 - k)$ as follows. As $s_k = s[|s| - (p_i - p + \triangle) - (|r| - p_k), l_k]$, we have $k \in [\tau + 1 - (p_i - p + \triangle) + 1, \tau + 1]$.

$$\begin{aligned}
p_k + \triangle - (\tau + 1 - k) &= p_k + (|s| - |r|) - (\tau + 1) + k \\
&\ge p_k + (|s| - |r|) - (\tau + 1) + \\
&\qquad \tau + 1 - (p_i - p + \triangle) + 1 \\
&= p_k + (|s| - |r|) - (p_i - p + \triangle) + 1 \\
&> |s| - (p_i - p + \triangle) - (|r| - p_k)
\end{aligned}$$

Thus $s[|s| - (p_i - p + \triangle) - (|r| - p_k), l_k]$ will not match any substring in $\mathcal{W}_m(s, \mathcal{L}_l^k)$. That is $s_k \ne s'_k$.

Case 3: $s_k = s_m$ and $s'_k = s[|s|-(p_j-p'+\triangle)-(|r|-p_k), l_k]$. It is similar to Case (2);

Case 4: $s_k = s[p_k + (p - p_i), l_k]$ and $s'_k = s'_m$. In this case, $p > p_i$. We prove that $s_k \ne s'_k$ as follows. If $k \ne j$, $s_k \ne s'_k$, as s_k is selected for the k-th segment and s'_k is selected for j-th segment. If $k = j$, the start positions of $s'_k \in \mathcal{W}_m(s, \mathcal{L}_l^k)$ are in $[p_k + \triangle - (\tau + 1 - k), p_k + \triangle + (\tau + 1 - k)] \cap [p_k - (k - 1), p_k + (k - 1)]$. As $k \in [1, p - p_i]$, we have $p_k + (k - 1) \le p_k + (p - p_i) - 1 < p_k + (p - p_i)$. Thus $s[p_k+(p-p_i), l_k]$ will not match any substring in $\mathcal{W}_m(s, \mathcal{L}_l^k)$. That is $s_k \ne s'_k$.

Case 5: $s_k = s_m$ and $s'_k = s[p_k + (p' - p_j), l_k]$. It is similar to Case (4);

Case 6: $s_k = s[p_k + (p - p_i), l_k]$ and $s'_k = s[|s| - (p_j - p' + \triangle) - (|r| - p_k), l_k]$, that is $p > p_i$ and $p' < p_j + \triangle$. We prove that $s_k \ne s'_k$ as follows. Based on the proofs of Case 4 and Case 3, we have $p_k + (k - 1) < p_k + (p - p_i)$ and $|s|-(p_j-p'+\triangle)-(|r|-p_k) < p_k+\triangle-(\tau+1-k)$. Meanwhile as $\triangle \le \tau$, we have $p_k + \triangle - (\tau + 1 - k) \le p_k + (k - 1)$. Thus $s[p_k + (p - p_i), l_k] \ne s[|s| - (p_j - p' + \triangle) - (|r| - p_k), l_k]$ as their start positions are not the same. That is $s_k \ne s'_k$.

Case 7: $s_k = s[|s| - (p_i - p + \triangle) - (|r| - p_k), l_k]$ and $s'_k = s[p_k + (p' - p_j), l_k]$. It is similar to Case (6).

Case 8: $s_k = s[p_k+(p-p_i), l_k]$ and $s'_k = s[p_k+(p'-p_j), l_k]$, that is $p > p_i$ and $p' > p_j$. In this case, we consider the following two cases:

(1) $p-p_i \ne p'-p_j$: We have $p_k+(p-p_i) \ne p_k+(p'-p_j)$, thus $s_k \ne s'_k$.

(2) $p - p_i = p' - p_j$: We have $s_k = s'_k$ as $p_k + (p - p_i) = p_k+(p'-p_j)$ and both of their lengths are l_k. Next we prove $i \ne j$ by contradiction. Suppose $i = j$, we have $p_i = p_j$. As $p - p_i = p' - p_j$, $p = p'$. Thus $s_m = s'_m$ as they have the same start positions and the same lengths. This contradicts with $s_m \ne s'_m$. Thus $i \ne j$.

Then we prove that a method that only selects s_k (or s'_k) from $\Phi(s_m, l)$ and $\Phi(s_{m'}, l)$ will miss a similar pair. We construct a new string r' similar to the case of generating the string r using s_m except that (1) For the k-th substring of s, we apply an additional substitution operation (by substituting character $s[p_k + (p - p_i)]$ with a special character) such that the k-th segment of r' will not match any substring of s; and (2) The j-th segment of r' is exactly the j-th substring of s. Next we prove that the number of edit operations in the new transformation from s to r' is τ. Compared with the transformation $\mathcal{T}$ from s to r and the new transformation $\mathcal{T}'$ from s to r', $\mathcal{T}'$ has an additional substitution operation on the k-th substring and a match operation on the the j-th substring of s. Next we prove that in transformation $\mathcal{T}$, we do a substitution operation on the j-th substring.

As $s'_m \in \mathcal{W}_m(s, \mathcal{L}_l^j)$, we have $p' \in [p_j - (j-1), p_j + (j-1)] \cap [p_j + \triangle - (\tau+1-j), p_j + \triangle + (\tau+1-j)]$. Thus $p' - p_j < j < \tau + 1 - |p_j - p' + \triangle| + 1$. In addition, as $p - p_i = p' - p_j$, we have $\tau + 1 - |p_j - p' + \triangle| + 1 = \tau + 1 - |p_i - p + \triangle| + 1$. Thus $p - p_i < j < \tau + 1 - |p_i - p + \triangle| + 1$. As $i \neq j$, in the transformation $\mathcal{T}$, we do a substitution operation on the j-th substring.

Thus $\mathcal{T}$ and $\mathcal{T}'$ have the same number of edit operations, that is the number of edit operations in $\mathcal{T}'$ is τ.

Note that in the transformation $\mathcal{T}'$ from s to r', we only have the following match operations: (1) the substring s_m matches the i-th segment; (2) the j-th substring matches the j-th segment; and (3) the other substrings in $\Phi(s_m, l)$ and $\Phi(s_{m'}, l)$ except s_k match some other segments of r'. Next we prove that s_k is neither s_m nor the j-th substring.

Firstly as s_m is a substring for the i-th segment and s_k is a substring for the k-th segment ($k \neq i$), $s_k \neq s_m$.

Secondly $s'_k \neq s'_m$ as they are selected for different segments. Thus $s_k = s'_k \neq s'_m$. Next we prove that the j-th substring of s is exactly s'_m. Thus s_k is not the j-th substring of s. We prove it as follows. The start position of the j-th substring of s is $p_j + (p - p_i)$ as we do $p - p_i$ deletion operations in the transformation $\mathcal{T}$. In addition the start position of s'_m is p'. As $p_j + (p - p_i) = p_j + (p' - p_j) = p'$, the j-th substring and s'_m have the same start positions. On the other hand they have the same length. Thus the j-th substring in the transformation $\mathcal{T}$ is exactly s'_m.

As s_k does not match a segment of r', if we only select s_k, we will miss the similar pair $\langle s, r' \rangle$. Thus we cannot only select s_k (or s'_k) from $\Phi(s_m, l)$ and $\Phi(s_{m'}, l)$.

Case 9: $s_k = s[|s| - (p_i - p + \triangle) - (|r| - p_k), l_k]$ and $s'_k = s[|s| - (p_j - p' + \triangle) - (|r| - p_k), l_k]$. It is similar to Case (8).

Thus the lemma is proved. $\square$

Based on the two lemmas, next we prove that the substring set $\mathcal{W}_m(s, l)$ generated by the multi-match-aware selection method has the minimum size.

THEOREM 3. *The substring set $\mathcal{W}_m(s, l)$ generated by the multi-match-aware selection method has the minimum size among all the substring sets generated by the substring selection methods that satisfy completeness.*

PROOF. Consider any substring selection method satisfying completeness. For each $s_m \in \mathcal{W}_m(s, l)$, based on Lemma 3.1 the method must select a substring in $\Phi(s_m, l)$. Based on Lemma 3.2, for different substrings s_m and s'_m in $\mathcal{W}_m(s, l)$, the method must select two different substrings. Thus the method must select $|\mathcal{W}_m(s, l)|$ substrings, and the theorem is proved. $\square$

G. PROOF OF THEOREM 4

THEOREM 4. *If $l \geq 2(\tau+1)$ and $|s| \geq l$, $\mathcal{W}_m(s, l)$ satisfies minimality.*

PROOF. If $l \geq 2(\tau + 1)$, the substrings with deletion operations must contain at least 3 characters. In this case, $\Phi(s_m, l) = \{s_m\}$, thus any substring selection method must select $\{s_m\}$ based on Theorem 3. Thus $\mathcal{W}_m(s, l)$ satisfies minimality. $\square$

H. PROOF OF LEMMA 4

LEMMA 4. *Given strings s and r, if each value in $E(i, *)$ is larger than τ, the edit distance of r and s is larger than τ.*

PROOF. We prove that any transformation from r to s will involve more than τ edit operations if each value in $E(i, *)$ is larger than τ. For any transformation $\mathcal{T}$ from r to s, $\mathcal{T}$ will must include one of $M(i, *)$. Without loss of generality, suppose $\mathcal{T}$ includes $M(i, j)$. Then we have $d_1 \geq M(i, j)$ and $d_2 \geq \big|(|s| - j) - (|r| - i)\big|$. Thus $|\mathcal{T}| = d_1 + d_2 \geq M(i, j) + \big|(|s| - j) - (|r| - i)\big| = E(i, j) > \tau$. Thus transformation $\mathcal{T}$ will involve more than τ edit operations. Therefor the edit distance of r and s is larger that τ. $\square$

I. PROOF OF THEOREM 5

Given a string r and a string s, s has a substring s_m matching a segment r_m of r. If $\langle s, r \rangle$ passes our verification algorithm, that is $\text{ED}(r_l, s_l) + \text{ED}(r_r, s_r) \leq \tau$, $\langle s, r \rangle$ must be a similar pair as $\text{ED}(r, s) \leq \text{ED}(r_l, s_l) + \text{ED}(r_r, s_r) \leq \tau$. Thus our extension-based method satisfies condition (1). To prove condition (2), we need to prove that s must have a substring s_m which matches a segment r_m of r such that $s_m \in \mathcal{W}_m(s, l)$ and $\text{ED}(r_l, s_l) + \text{ED}(r_r, s_r) = \text{ED}(r, s) \leq \tau$.

LEMMA 5.1. *If s is similar to r, s must have a substring s_m which matches a segment r_m of r, such that $s_m \in \mathcal{W}_m(s, l)$ and $\text{ED}(r, s) = \text{ED}(r_l, s_l) + \text{ED}(r_r, s_r)$.*

PROOF. We first prove that given a transformation $\mathcal{T}$ from r to s with $|\mathcal{T}| = \text{ED}(r, s)$ edit operations, for any segment r_m of r matching a substring s_m of s in $\mathcal{T}$, $\text{ED}(r, s) = \text{ED}(r_l, s_l) + \text{ED}(r_r, s_r)$. As $\mathcal{T}$ transforms r_l to s_l, matches r_m with s_m, and transforms r_r to s_r, we have $|\mathcal{T}| \geq \text{ED}(r_l, s_l) + \text{ED}(r_m, s_m) + \text{ED}(r_r, s_r)$. In addition, based on the definition of edit distance, we have $\text{ED}(r_l, s_l) + \text{ED}(r_m, s_m) + \text{ED}(r_r, s_r) \geq \text{ED}(r, s)$. As $|\mathcal{T}| = \text{ED}(r, s)$ and $\text{ED}(r_m, s_m) = 0$, $\text{ED}(r, s) = \text{ED}(r_l, s_l) + \text{ED}(r_r, s_r)$.

Then, based on Theorem 2, for any transformation $\mathcal{T}$, there must exist a substring $s_m \in \mathcal{W}_m(s, l)$ of s that matches a segment r_m of r, and we have $\text{ED}(r, s) = \text{ED}(r_l, s_l) + \text{ED}(r_r, s_r)$. Thus the lemma is proved. $\square$

Then based on Lemma 5.1, we prove Theorem 5.

THEOREM 5. *Our extension-based verification method satisfies correctness.*

PROOF. We first prove the condition (1). If $\langle s, r \rangle$ passes our verification algorithm, then there exists a segment r_m matching a substring s_m and $\text{ED}(s_l, r_l) + \text{ED}(s_r, r_r) \leq \tau$. Thus there exists a transformation from r to s with no large than τ edit distance. Hence $\langle s, r \rangle$ must be a similar pair.

Then we prove the condition (2). If $\langle s, r \rangle$ is a similar pair, any transformation $\mathcal{T}$ with $|\mathcal{T}| = \text{ED}(r, s) \leq \tau$ edit operations will have a segment r_m matching a substring s_m of s. Based on Lemma 5.1, we have $\text{ED}(s, r) = \text{ED}(s_l, r_l) + \text{ED}(s_r, r_r) \leq \tau$. As $\text{ED}(s_l, r_l) + \text{ED}(s_r, r_r) \leq \tau$, $\langle s, r \rangle$ must pass our verification algorithm. $\square$

J. PROOF OF THEOREM 6

Based on completeness of our multi-match-aware selection method and correctness of our extension-based verification method, next we prove correctness and completeness of our algorithms.

THEOREM 6. *Our algorithm satisfies (1) completeness: Given any similar pair $\langle s, r\rangle$, our algorithm must find it as an answer; and (2) correctness: A pair $\langle s, r\rangle$ found in our algorithm must be a similar pair.*

PROOF. We first prove completeness of our method. That is given a similar pair $\langle s, r\rangle$, our method must find it as an answer. Without loss of generality, suppose r is visited before s. Based on Theorem 2, our multi-match-based method must find this pair as a candidate pair. Based on Theorem 5, the similar pair $\langle s, r\rangle$ can pass our extension-based verification, and thus it must be added as an answer. Thus our algorithm satisfies completeness.

Then we prove correctness of our method. That is a pair $\langle s, r\rangle$ found in our algorithm must be a similar pair. Based on Theorem 5, any pair $\langle s, r\rangle$ passed our extension-based verification must be a similar pair. Thus our algorithm satisfies correctness. $\square$